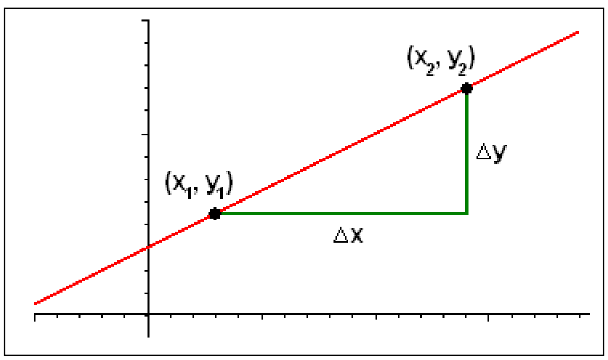
MATH 166
Lesson 1.4
Two Problems with One Theme

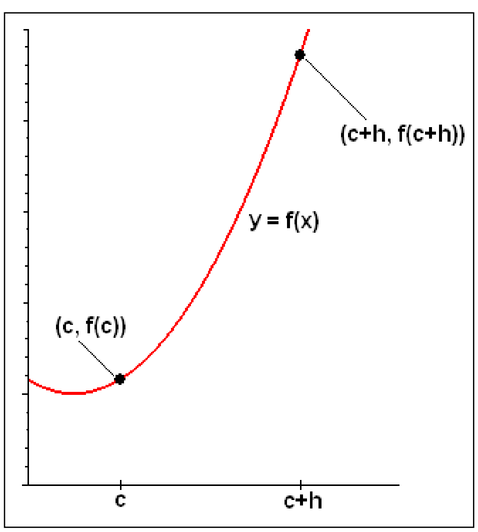
In this section we will discuss two specific problems. One of the problems is strictly geometric; the other has practical value in the physical world. By lesson's end, we'll see that the problems are one in the same.

Problem 1: The Tangent Line

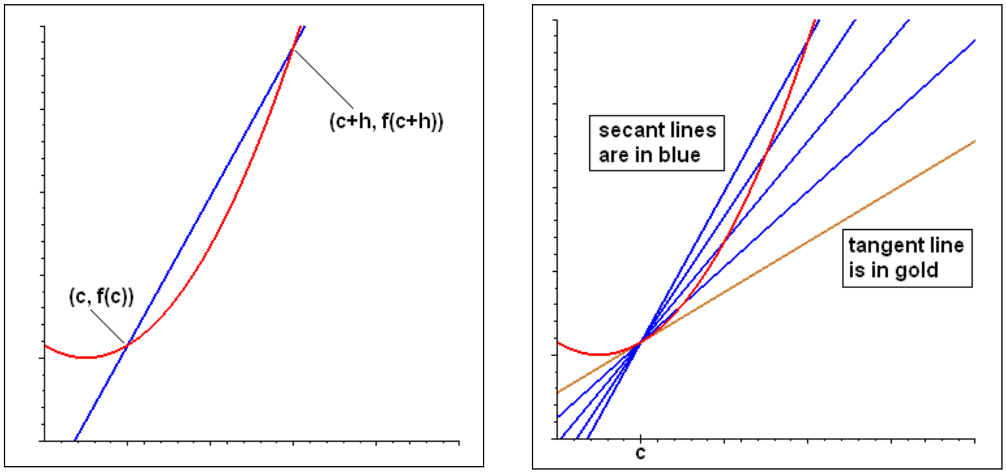
First recall how to compute the slope of a line. Given two points on a line, say (x_1, y_1) and (x_2, y_2), the slope of the line is given by $\frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$. See the picture below.



The letter *m* is usually reserved for slope so we may write $m = \frac{y_2 - y_1}{x_2 - x_1}$. This idea extends easily to the slope of a *curve*. First, look at the picture below. It considers two points on the graph of $y = f(x)$—namely $(c, f(c))$ and $(c+h, f(c+h))$.



Now draw a line that connects these points. See below (left).



This line (in blue) is called a **secant line**. The slope of this secant line is given by $\frac{\Delta y}{\Delta x} = \frac{f(c+h) - f(c)}{(c+h) - c} = \frac{f(c+h) - f(c)}{h}$. It is common to write $m_{\sec} = \frac{f(c+h) - f(c)}{h}$. If there is a limiting position for this line as *h* shrinks (that is, *h* gets closer and closer to 0), then the result is called a **tangent line**. Most tangent lines just "touch" the function at the point $(c, f(c))$. Look carefully at the picture (see diagram, above right). The two points $(c, f(c))$ and $(c+h, f(c+h))$ become one. With this idea, we can introduce the *slope* of this tangent line.

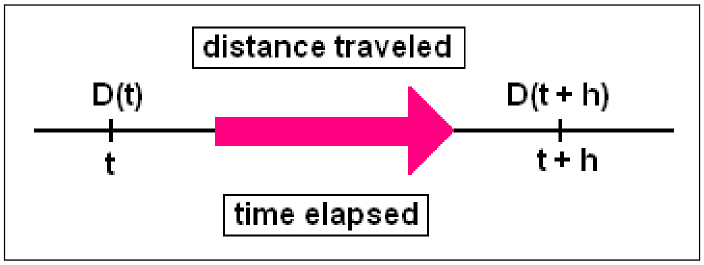
Definition: The tangent line to the curve $y = f(x)$ at the point $(c, f(c))$ is the line through $(c, f(c))$ that has slope

$$m_{\tan} = \lim_{h \to 0} m_{\sec} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

Note: We interpret the above notation "$\lim_{h \to 0}$" as the "limit as *h* approaches zero." In other words, we let *h* tend to zero. We make this more precise later on!

Problem 2: Instantaneous Velocity

We now begin our study of a (seemingly) different topic. You may recall the formula Distance = (Rate)(Time), often abbreviated as $D = RT$. If you solve for the rate, you get $R = \frac{D}{T}$. It is this formula that we will study more closely. If we let *t* stand for time and *D* stand for distance traveled, we can see how far we have gone after *h* units of time have passed:



A rate can be computed by $R = \frac{D}{T} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{D(t+h)-D(t)}{h}$. Since this rate takes into account the entire time interval, it is an average rate. This is typically called **average velocity**. Your authors may write something like $v_{\text{avg}} = \frac{f(c+h)-f(c)}{h}$ with distance function $f(t)$. This is exactly what we derived above with nothing more than a change in notation. When you take a long trip, you are undoubtedly interested in your average velocity or average speed. However, at any point during the trip, your speedometer doesn't report this average (necessarily). Instead, it reports your speed <u>at the very instant that you glance at it</u>. This is defined as **instantaneous velocity**. Just like in our first problem, we let $h \to 0$ and this gives us a precise reading of our velocity at this exact moment.

<u>Definition</u>: If an object moves along a line with position function $f(t)$, then its **instantaneous velocity** at time $t = c$ is given by

$$v = \lim_{h\to 0} v_{\text{avg}} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h},$$

where the limit notation is interpreted similarly to beforehand.

Quite surprisingly, you should see that Problems 1 and 2 are one in the same. Given a function that indicates an object's position, finding the **slope of the tangent line** to the graph of the function at the point $x = c$ is *precisely the same* as finding the **instantaneous velocity** of the object at time $t = c$!