MATH 166
Lesson 1.5a
Limit of a Function

Here, we will start by looking at this problem:

What is the behavior of $f(x) = \dfrac{x^3 - 2x^2 + 3x - 6}{x - 2}$ as *x* approaches 2?

A table of values "near" $x = 2$ would generate something like the screen below (we'll do this in class):

X	Y1	
1.97	6.8809	
1.98	6.9204	
1.99	6.9601	
2	ERROR	
2.01	7.0401	
2.02	7.0804	
2.03	7.1209	
X=1.97		

Notice that all the numbers in the Y column are "near" 7 (and this is even more pronounced for the *x*'s closest to 2). That is, it appears that $f(x)$ *is approaching 7 as x approaches 2*. $f(x)$ never reaches 7 but that doesn't matter. Does it get close to 7? It sure does! We could "zoom in" closer to 2 and get the following:

X	Y1	
1.997	6.988	
1.998	6.992	
1.999	6.996	
2	ERROR	
2.001	7.004	
2.002	7.008	
2.003	7.012	
X=1.997		

Why the "error" when $x = 2$? If you attempt to find $f(2)$, you will get

$f(2) = \dfrac{2^3 - 2(2^2) + 3(2) - 6}{2 - 2} = \dfrac{0}{0}$ which is undefined! No wonder the table has problems reporting a value when $x = 2$. This example should convince you of one important fact: **In order to examine a function's behavior near a point, it is <u>not sufficient</u> to simply investigate the function's <u>value</u> at that point.**

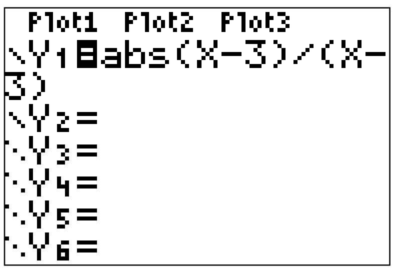
Based on the discussion from above, we can say "as *x* gets close to 2, $f(x)$ gets close to 7." In symbols, we write: as $x \to 2$, $f(x) \to 7$. The most popular way to express this is via the compact statement $\lim_{x\to 2} f(x) = 7$. This is read, "The limit of $f(x)$ as x approaches 2 is 7." Here is the general statement.

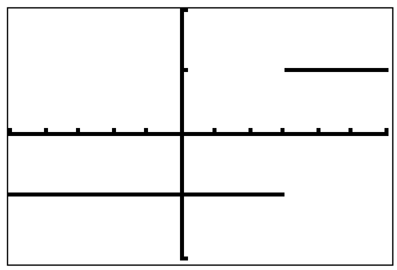
We write $\lim_{x\to c} f(x) = L$ if, as x approaches c, $f(x)$ gets arbitrarily close to L.

Notice that this statement mentions nothing about the specific value $f(c)$. *What happens at* $x = c$ *is irrelevant.*

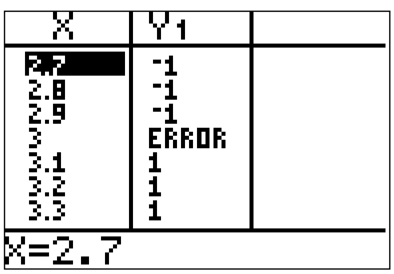
<u>Example</u>: Find $\lim_{x\to 3} \frac{|x-3|}{x-3}$.

<u>Solution</u>: Much like the previous example, "plugging in" $x = 3$ fails so we will look at a graph or a table. Why not look at both? Here we see the graph….





…and here is a table.



X	Y1
2.7	-1
2.8	-1
2.9	-1
3	ERROR
3.1	1
3.2	1
3.3	1

X=2.7

Hmmmmm. It seems that from the left-hand side of 3 (numbers less than 3), the limit is -1. In fact, the function doesn't seem to be *approaching* -1; it's just stuck there! On the contrary, from the right-hand side of 3 (numbers greater than 3), the limit is 1. **Look at the graph and convince yourself that you could have extracted this same information from the picture.** So as x approaches 3, what does $\frac{|x-3|}{x-3}$ approach? Unfortunately, the function approaches *different* values, depending on the direction from which we are traveling (that is, from the right or from the left). Because the function does not approach <u>any one number</u>, we say that $\lim_{x\to 3} \frac{|x-3|}{x-3}$ *does not exist*, or DNE for short.

The example above illustrates the concept of a **one-sided limit**. If we are traveling from the left of *c*, then we write $x \to c^-$; if traveling from the right of *c*, we write $x \to c^+$. In the previous example, we could have written

$$\boxed{\lim_{x \to 3^-} \frac{|x-3|}{x-3} = -1 \text{ and } \lim_{x \to 3^+} \frac{|x-3|}{x-3} = 1.}$$

Since the one-sided limits are different, this is why $\lim_{x \to 3} \frac{|x-3|}{x-3}$ fails to exist. This is the chief reason we look on <u>both</u> sides of $x = c$ when examining a table.