

MATH 166
Lesson 1.5a
Limit of a Function

Here, we will start by looking at this problem:

What is the behavior of $f(x) = \frac{x^3 - 2x^2 + 3x - 6}{x - 2}$ as x approaches 2?

A table of values “near” $x = 2$ would generate something like the screen below (we’ll do this in class):

X	Y1	
1.97	6.8809	
1.98	6.9204	
1.99	6.9601	
2	ERROR	
2.01	7.0401	
2.02	7.0804	
2.03	7.1209	
X=1.97		

Notice that all the numbers in the Y column are “near” 7 (and this is even more pronounced for the x ’s closest to 2). That is, it appears that $f(x)$ is *approaching 7 as x approaches 2*. $f(x)$ never reaches 7 but that doesn’t matter. Does it get close to 7? It sure does! We could “zoom in” closer to 2 and get the following:

X	Y1	
1.997	6.988	
1.998	6.992	
1.999	6.996	
2	ERROR	
2.001	7.004	
2.002	7.008	
2.003	7.012	
X=1.997		

Why the “error” when $x = 2$? If you attempt to find $f(2)$, you will get

$$f(2) = \frac{2^3 - 2(2^2) + 3(2) - 6}{2 - 2} = \frac{0}{0}$$

which is undefined! No wonder the table has problems

reporting a value when $x = 2$. This example should convince you of one important fact: **In order to examine a function’s behavior near a point, it is not sufficient to simply investigate the function’s value at that point.**

Based on the discussion from above, we can say “as x gets close to 2, $f(x)$ gets close to 7.” In symbols, we write: as $x \rightarrow 2$, $f(x) \rightarrow 7$. The most popular way to

express this is via the compact statement $\lim_{x \rightarrow 2} f(x) = 7$. This is read, “The limit of $f(x)$ as x approaches 2 is 7.” Here is the general statement.

We write $\lim_{x \rightarrow c} f(x) = L$ if, as x approaches c , $f(x)$ gets arbitrarily close to L .

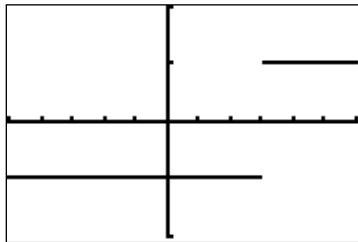
Notice that this statement mentions nothing about the specific value $f(c)$. *What happens at $x = c$ is irrelevant.*

Example: Find $\lim_{x \rightarrow 3} \frac{|x-3|}{x-3}$.

Solution: Much like the previous example, “plugging in” $x = 3$ fails so we will look at a graph or a table. Why not look at both?

```

Plot1 Plot2 Plot3
\Y1=abs(X-3)/(X-
3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
    
```



X	Y1
2.7	-1
2.8	-1
2.9	-1
3	ERROR
3.1	1
3.2	1
3.3	1

X=2.7

Hmmmm. It seems that from the left-hand side of 3 (numbers less than 3), the limit is -1 . In fact, the function doesn't seem to be *approaching* -1 ; it's just stuck there! On the contrary, from the right-hand side of 3 (numbers greater than 3), the limit is 1 . **Look at the graph and convince yourself that you could have extracted this same**

information from the picture. So as x approaches 3, what does $\frac{|x-3|}{x-3}$ approach?

Unfortunately, the function approaches *different* values, depending on the direction from which we are traveling (that is, from the right or from the left). Because the function does not approach any one number, we say that $\lim_{x \rightarrow 3} \frac{|x-3|}{x-3}$ *does not exist*, or DNE for short.

The example above illustrates the concept of a **one-sided limit**. If we are traveling from the left of c , then we write $x \rightarrow c^-$; if traveling from the right of c , we write $x \rightarrow c^+$. In the previous example, we could have written

$$\lim_{x \rightarrow 3^-} \frac{|x-3|}{x-3} = -1 \text{ and } \lim_{x \rightarrow 3^+} \frac{|x-3|}{x-3} = 1.$$

Since the one-sided limits are different, this is why $\lim_{x \rightarrow 3} \frac{|x-3|}{x-3}$ fails to exist. This is the chief reason we look on both sides of $x = c$ when examining a table.