MATH 166
Lesson 1.6a
Using Limit Laws

I think you will like this section. We won't necessarily abandon the tools we've acquired (checking graphs, analyzing tables, etc.) but we'll learn *much better* ways of finding limits. First, here is the main theorem of the section.

Theorem: Let *n* be a positive integer, *k* be any number, and $f(x)$ and $g(x)$ be functions that have limits as $x \to c$. Then the following statements are true.

(a) $\lim\limits_{x \to c} k = k$

(b) $\lim\limits_{x \to c} x = c$

(c) $\lim\limits_{x \to c} \left[k \cdot f(x)\right] = k \cdot \lim\limits_{x \to c} f(x)$

(d) $\lim\limits_{x \to c} \left[f(x) \pm g(x)\right] = \lim\limits_{x \to c} f(x) \pm \lim\limits_{x \to c} g(x)$

(e) $\lim\limits_{x \to c} \left[f(x) \cdot g(x)\right] = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x)$

(f) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$, provided that $\lim\limits_{x \to c} g(x) \neq 0$

(g) $\lim\limits_{x \to c} \left[f(x)\right]^n = \left[\lim\limits_{x \to c} f(x)\right]^n$

(h) $\lim\limits_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to c} f(x)}$, provided that $\lim\limits_{x \to c} f(x) > 0$ for *n* even

This may look like a lot to swallow but it basically says "just plug in" if it works. So we have already seen this! A far more important result follows.

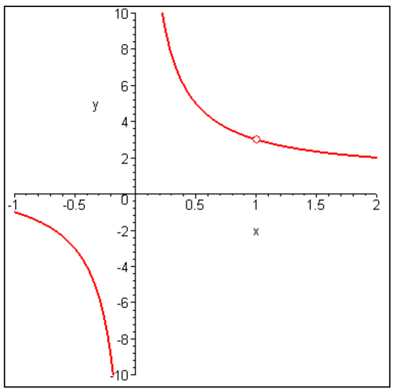
Theorem: If $f(x) = g(x)$ for all *x* in an open interval containing the number *c*, except possibly at the number *c* itself, and if $\lim\limits_{x \to c} g(x)$ exists, then $\lim\limits_{x \to c} f(x)$ exists. Furthermore, the limits are equal.

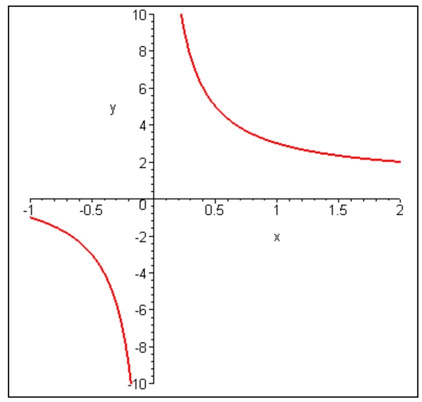
Phrased differently, if two functions differ by only a single point but one of the functions has a limit at that point, then so does the other function (and the limits are identical).

This helps us out a great deal with problems such as $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - x}$.

Example: Find $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - x}$.

<u>Solution</u>: First notice that $\dfrac{x^2+x-2}{x^2-x}=\dfrac{(x+2)(x-1)}{x(x-1)}=\dfrac{x+2}{x}$. Call the leftmost expression $y_1=\dfrac{x^2+x-2}{x^2-x}$ and the rightmost expression $y_2=\dfrac{x+2}{x}$. Although they are linked together by equal signs, they are <u>not</u> always equal!! In particular, $y_1(1)$ is not defined but $y_2(1)=3$. **Everywhere else the functions agree.** Look at the graphs below and compare.





$y_1=\dfrac{x^2+x-2}{x^2-x}$ $\qquad$ $y_2=\dfrac{x+2}{x}$

Now apply the theorem above so that

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{(x+2)(x-1)}{x(x-1)}=\lim_{x\to 1}\frac{x+2}{x}=\frac{1+2}{1}=3.$$

The above technique is common for problems of this type. If you see a cancellation, then go for it.

<u>Example</u>: Find $\lim\limits_{h\to 0}\dfrac{\sqrt{2+h}-\sqrt{2}}{h}$.

<u>Solution</u>: Again, direct substitution yields $\dfrac{0}{0}$ so let's try some algebra. There is no obvious cancellation here so we will try a common algebra trick; multiply numerator and denominator by the conjugate expression $\sqrt{2+h}+\sqrt{2}$ (the same expression that you see in the numerator but with the opposite sign). So we have

$$\lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h}=\lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h}\cdot\frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}$$

$$= \lim_{h \to 0} \frac{(2+h) + \sqrt{2}\sqrt{2+h} - \sqrt{2}\sqrt{2+h} - 2}{h\left(\sqrt{2+h} + \sqrt{2}\right)}$$

$$= \lim_{h \to 0} \frac{(2+h) - 2}{h\left(\sqrt{2+h} + \sqrt{2}\right)}$$

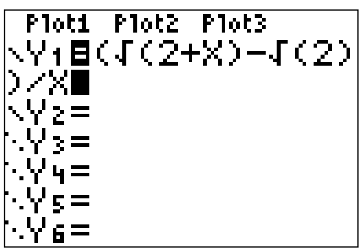
$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{2+h} + \sqrt{2}\right)}$$

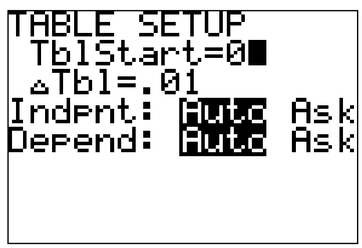
$$= \lim_{h \to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}}$$

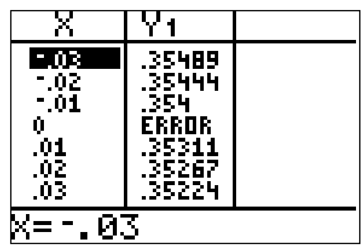
$$= \frac{1}{\sqrt{2+0} + \sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}}. \qquad \leftarrow \text{Answer}$$

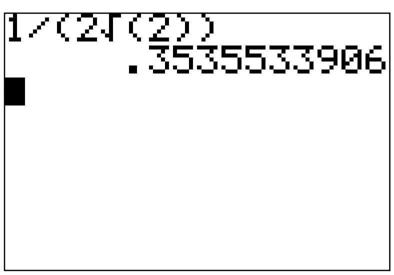
Notice the FOIL method was used in step 2, and "just plug in" was used in step 6. You can confirm this limit by calculator.







Of course, this number 0.35 (or so) is rather mysterious. If you go back to the home screen and check $\frac{1}{2\sqrt{2}}$, everything looks good.



You may notice that the algebraic method illustrated here is far superior to using a table. After all, we know the *exact* answer is $\frac{1}{2\sqrt{2}}$ without even looking at a graph or a table. On the other hand, the table only gives an approximation.