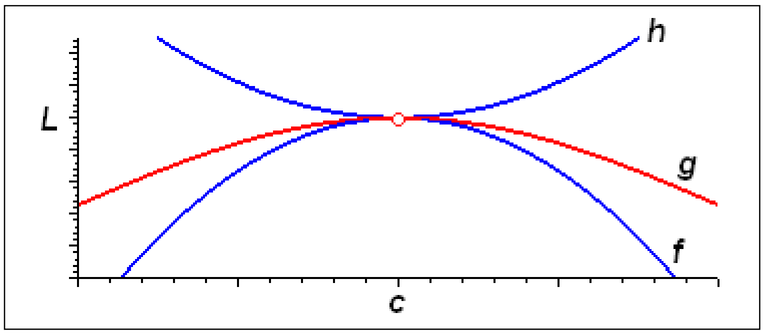
MATH 166
Lesson 1.6b
Limit Laws/Squeeze Theorem

Here is the main act for this section:

> Squeeze Theorem: Let $f(x)$, $g(x)$, and $h(x)$ satisfy the inequality $f(x) \le g(x) \le h(x)$ for all *x* near *c* except possibly at *c*. If $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.

Look at the picture below. "$g(x)$ is squeezed between $f(x)$ and $h(x)$."



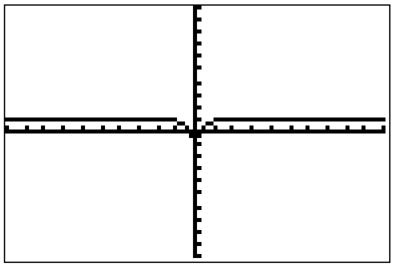
Here is an example of the squeeze theorem in action.

Example: Show that $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$ by using the squeeze theorem. *Hint*: Use the functions $y = |x|$ and $y = -|x|$ to assist in your solution.

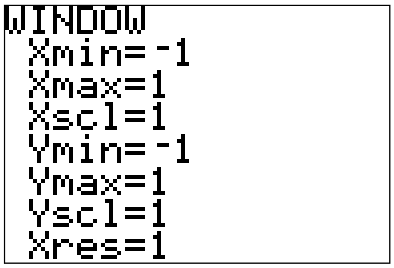
Solution: First, notice that an **incorrect** application of the direct substitution theorem would give

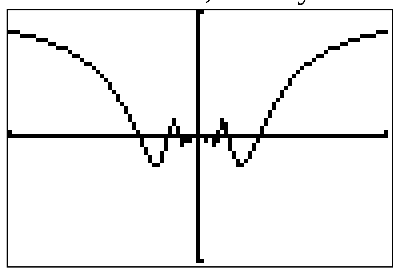
$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 = \lim_{x \to 0} x \cdot \lim_{x \to 0} \sin\left(\frac{1}{x}\right).$$

This is invalid since the second limit, $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$, fails to exist (remember the oscillation with this graph?). So let us take a look at the graph of $y = x \sin\left(\frac{1}{x}\right)$ in the standard window $[-10, 10, -10, 10]$:

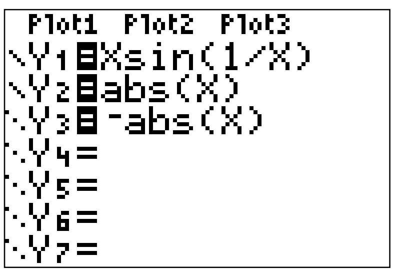


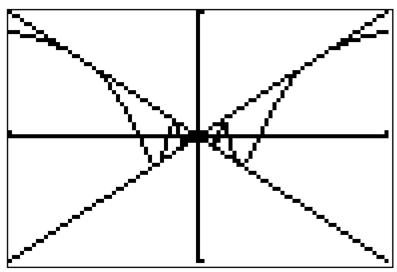
Since we are concerned with behavior near $x=0$, modify the window to reflect this:





Now sketch the functions $y=|x|$ and $y=-|x|$ in the same window.





From the picture, you can deduce that $-|x| \le x\sin\left(\frac{1}{x}\right) \le |x|$. In other words, $y=|x|$ is the top function, $y=x\sin\left(\frac{1}{x}\right)$ is the middle function, and $y=-|x|$ is the bottom function. Now if you let $x \to 0$, both $|x| \to 0$ and $-|x| \to 0$. That is, $\lim\limits_{x\to 0}|x|=0$ and $\lim\limits_{x\to 0}\left(-|x|\right)=0$. Since $y=x\sin\left(\frac{1}{x}\right)$ is "squeezed" between these two functions, it must be that $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right)=0$ as well. The zoomed in graph illustrates this well: