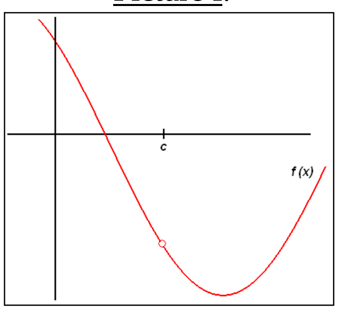
MATH 166
Lesson 1.8a
Continuity of Functions

Question: What does it mean to say that a function is *continuous*?
Answer: From an informal point of view, you can sketch the graph of this function without lifting your pencil from the paper.

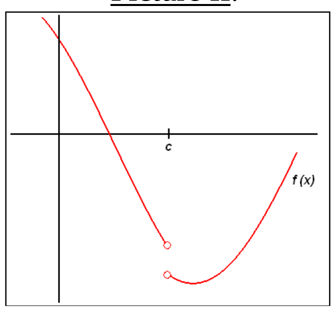
Most people would agree with the informal definition above. Continuous functions have no holes, breaks, or gaps. However, as we learned from our discussions on limits, this definition of "continuous" will have to be made more precise. We begin by asking, "What do <u>dis</u>continuous functions look like?"

<u>Investigation</u>: Look at the following three pictures and try to answer this question: WHY IS THE FUNCTION $y = f(x)$ DISCONTINUOUS? Give this some thought before you look at the answers; otherwise you won't benefit much from this exercise.

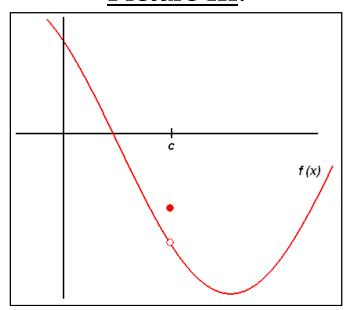
<u>Picture I</u>:



<u>Picture II</u>:



<u>Picture III</u>:



<u>Solution</u>: Why are the functions not continuous?

<u>Picture I</u>: $f(c)$ is not defined. The function has a hole at $x = c$.

<u>Picture II</u>: $\lim_{x \to c} f(x)$ does not exist. Although $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ both exist, they are not the same. The graph "jumps" at $x = c$.

<u>Picture III</u>: $\lim_{x \to c} f(x) \neq f(c)$. Even though $\lim_{x \to c} f(x)$ exists and $f(c)$ is defined, they are not the same.

This investigation strongly suggests a definition for continuity at a point $x = c$. Specifically, negate the boxed statements above.

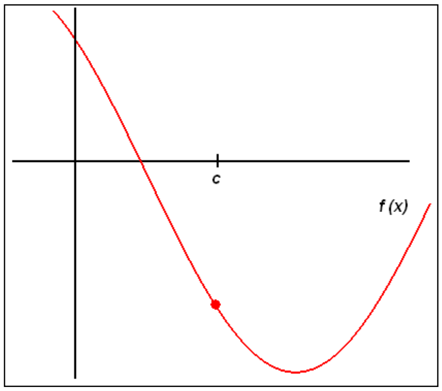
<u>Definition</u>: The function $y = f(x)$ is said to be **continuous** at $x = c$ provided that

(1) $f(c)$ is defined,

(2) $\lim_{x \to c} f(x)$ exists, and

(3) $\lim_{x \to c} f(x) = f(c)$.

Then the picture looks like



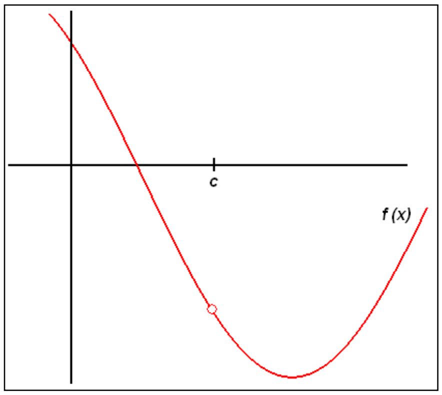
Convince yourself that all three parts of the definition are satisfied.

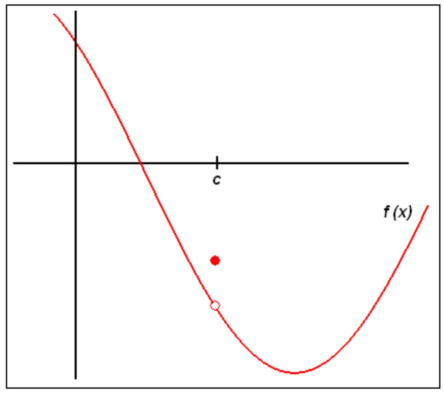
Note: Some authors say much less when defining continuity. For example, some textbooks says that $y = f(x)$ is **continuous** at $x = c$ if $\lim_{x \to c} f(x) = f(c)$. Specifically, it seems as if their definition only considers part (3) of our definition. However, notice that the minute you write down $\lim_{x \to c} f(x) = f(c)$, you are inherently *assuming* both (1) and (2). Thus, the definitions are equivalent. Personally, I am a big fan of the three-part definition since it breaks things down clearly.

It should be clear that polynomials are continuous at every real number *c*. You may say that polynomials are *everywhere continuous*. In a similar way, rational functions and trigonometric functions are continuous where they are defined. More precisely, these functions are continuous at all points in the interior of their corresponding domains.

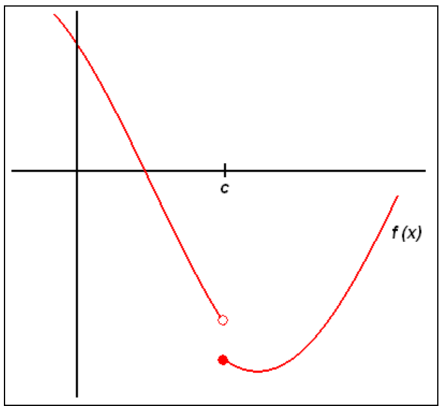
From the opening investigation, it should make sense that if any of the statements (1), (2), or (3) fail to hold, the result is a **discontinuous** function. There are two types of discontinuities.

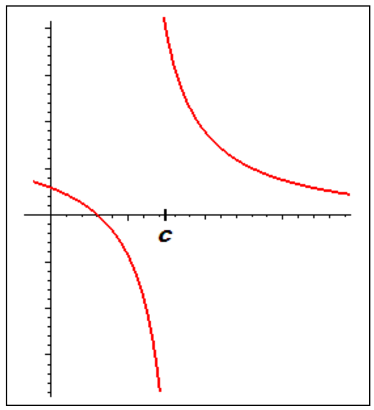
1. **Removable discontinuities**: The function $y = f(x)$ can be made continuous simply by "adjusting" one point. The pictures that follow illustrate removable discontinuities at $x = c$.





2. **Nonremovable discontinuities**: Just by looking at the name, these functions <u>cannot</u> be made continuous by "adjusting" one point. Look at the pictures below—both have nonremovable discontinuities at $x = c$.





To elaborate, the graph on the left cannot be made continuous just by redefining the function at one point. You would have to physically lift and move an entire branch of the function in order to make this function continuous.