MATH 166
Lesson 1.8b
Intermediate Value Theorem

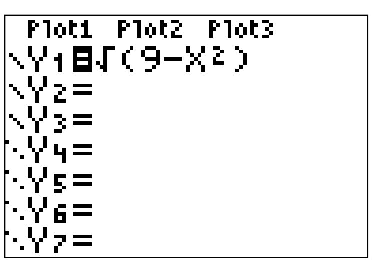
The discerning student will notice that we always use *open* intervals when discussing continuity. For example, we can say things such as $f(x)=x^2$ is continuous on $(3,5)$, $g(x)=\dfrac{1}{x}$ is continuous on $(0,2)$, $h(x)=\sqrt{x}$ is continuous on $(0,1)$, etc. The final concept discussed in this section is continuity on a *closed* interval. This ultimately leads to a very important theorem in calculus.

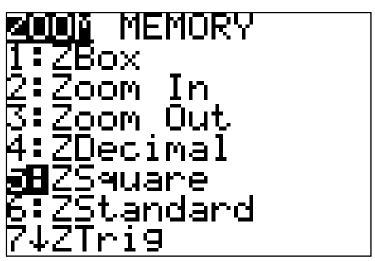
<u>Definition</u>: A function $y=f(x)$ is **right continuous** at $x=a$ if $\lim\limits_{x\to a^+} f(x)=f(a)$ and **left continuous** at $x=b$ if $\lim\limits_{x\to b^-} f(x)=f(b)$.

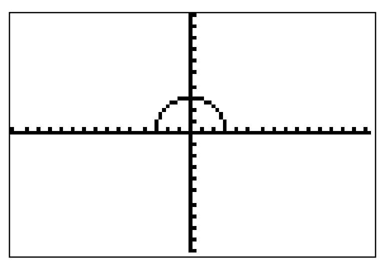
To pad the above definition, we already know that we can label a function "continuous on an open interval" if the function is continuous at each point in the interval. We can now add this: A function is **continuous on a closed interval** $[a,b]$ if the function is continuous on (a,b), right continuous at $x=a$, and left continuous at $x=b$. The following example should clarify.

<u>Example</u>: What is the largest interval for which $g(x)=\sqrt{9-x^2}$ is continuous?

<u>Solution</u>: First, you may recall that this graph is a semicircle. To see this, write $y=\sqrt{9-x^2}$, square both sides to get $y^2=9-x^2$, and rewrite in the standard form $x^2+y^2=9$. This represents a circle at the origin with radius 3. If you reverse steps and solve for y, you would obtain $y=\pm\sqrt{9-x^2}$. Since $g(x)=\sqrt{9-x^2}$ only takes the positive sign, this graph will only contain the *top half* of the circle. Let's graph this.







The domain of this function is $[-3,3]$. We can visually detect that the function is continuous on the interior of its domain; that is, $g(x)=\sqrt{9-x^2}$ is continuous on $(-3,3)$. But if we look at the one-sided limits we get

$$\boxed{\lim_{x\to -3^+}\sqrt{9-x^2}=\sqrt{9-9}=0=g(-3)}$$

and

$$\boxed{\lim_{x\to 3^-}\sqrt{9-x^2}=\sqrt{9-9}=0=g(3)}.$$

Glancing back at the previous definition, we may say that $g(x)$ is right continuous at $x=-3$ and left continuous at $x=3$. Hence, $g(x)=\sqrt{9-x^2}$ is continuous on the closed interval $[-3,3]$.

Here is the final result from this section. Not surprisingly, it uses continuity on a closed interval.

Intermediate Value Theorem: Let $y=f(x)$ be continuous on $[a,b]$. If k is any number between $f(a)$ and $f(b)$, then there must exist a number c in $[a,b]$ such that $f(c)=k$.

Here is a picture that illustrates this (Figure 1):

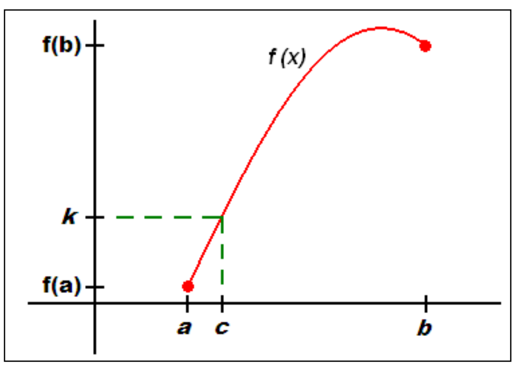


Figure 1.

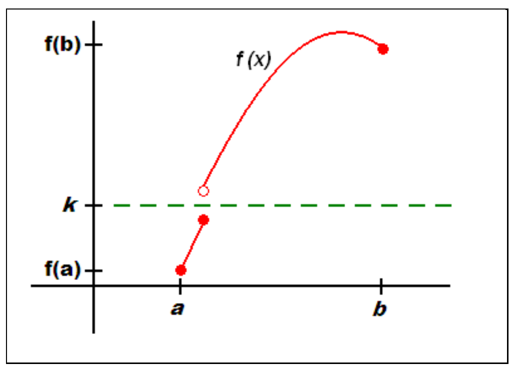


Figure 2.

Notice that continuity on $[a,b]$ is critical to this theorem. If we drop this assumption, the theorem is no longer true (Figure 2). In Figure 2, k is definitely between $f(a)$ and $f(b)$ but there is no c value in the interval $[a,b]$ where $f(c)=k$. This is a direct consequence of the break in the graph of $f(x)$; that is, the continuity assumption has been dropped.