**MATH 166**
**Lesson 2.2**
**Derivative as Function**

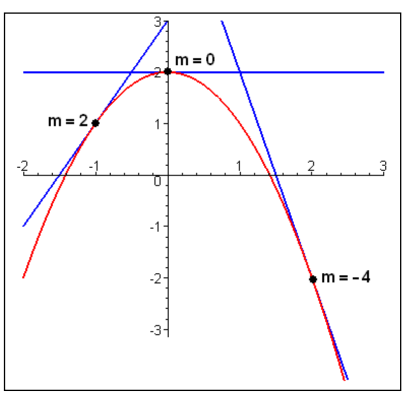
The star of this lesson is $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, the derivative.

Example: Find $g'(x)$ for $g(x)=2-x^2$.

Solution: Use the definition and take the limit at the end.

$$
\begin{aligned}
g'(x)&=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&=\lim_{h\to 0}\frac{\left(2-(x+h)^2\right)-\left(2-x^2\right)}{h}\\
&=\lim_{h\to 0}\frac{2-x^2-2hx-h^2-2+x^2}{h}\\
&=\lim_{h\to 0}\frac{h(-2x-h)}{h}\\
&=\lim_{h\to 0}(-2x-h)\\
&=-2x.
\end{aligned}
$$

Here, we get $g'(x)=-2x$. This problem illustrates that the derivative is yet another function (its meaning is still the slope of the tangent line). If you want to calculate the slope of the graph at different *x* values, you may do so. For example, $g'(-1)=2$, $g'(0)=0$ and $g'(2)=-4$. See the graphic below.




Being able to find a derivative (often referred to as *differentiability*) is a desirable characteristic for a function to have. However, not all functions will have derivatives at all points (the limit in the derivative definition may not exist). Some basic examples (with reasons) follow.

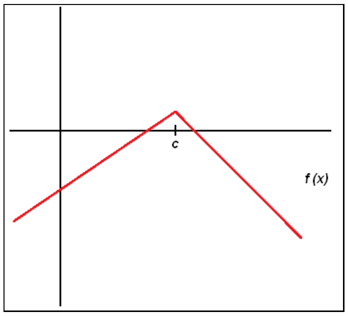



**Figure 1.**

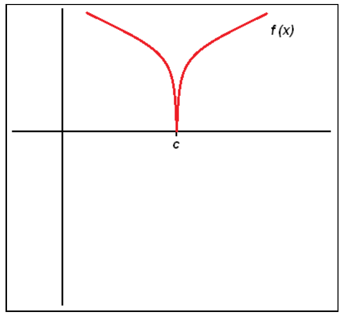



**Figure 2.**

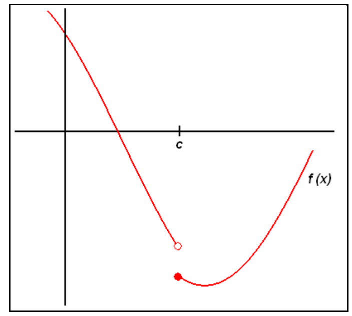



**Figure 3.**

**Figure 1:** This function does not have a derivative at $x = c$ (or you could say $f'(c)$ does not exist). If you attempt to calculate $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ at $x = c$, no such limit would exist. What you can visually detect is that the "slope" of the graph appears to be -1 on the right side of $x = c$ and +1 on the left of $x = c$. The abrupt change in the steepness at $x = c$ supports the impossibility of defining "slope" exactly when $x = c$.

**Figure 2:** You can use an argument similar to that in Figure 1 or you may see that the tangent lines become infinitely steep near $x = c$. As you get really close to $x = c$, the slope appears to grow in magnitude without bound. Since the slope cannot be calculated as any *finite* number, this is another case where we would say the function fails to have a derivative at $x = c$.

Note: Figures 1 and 2 suggest that an abrupt change/sharp turn in the graph leads to the function not being differentiable there. Rephrased, *differentiable functions are smooth*.

**Figure 3:** This function is discontinuous at $x = c$ so the idea of discussing slope at $x = c$ is inherently ambiguous. This leads inductively to the idea that if a function is discontinuous at a point, then it is not differentiable there. Logically equivalent to this is the statement, "if a function is differentiable at a point, then it must be continuous there." This, in fact, is a theorem.

<u>Theorem</u>: If $f(x)$ is differentiable at $x = c$ then $f(x)$ is continuous at $x = c$. In different terms, *differentiability implies continuity*.

Be sure you understand that the converse of the above statement is not true! That is, to say that "continuity implies differentiability" is a false claim! An example that immediately settles this is something like $f(x) = |x|$. We know that this function is perfectly continuous but $f'(0)$ fails to exist. Thus, just because a function is continuous at a point does not mean it is differentiable there.