MATH 166
Lesson 2.3b
Differentiation Rules (Product + Quotient)

Part two of this lesson discusses two more rules—one for the product of functions and one for the quotient. In light of what we have seen thus far, it may seem natural to think that $\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g'(x)$. Unfortunately, **this is not true**. A similar statement can be said for quotients. Let's take a look at an example with products to see why this doesn't work.

Example: Demonstrate that $\frac{d}{dx}\left[f(x)g(x)\right] \neq f'(x)g'(x)$ for the function $h(x) = x \cdot x^2$.

Solution: For $h(x) = x \cdot x^2$, we may surmise that the derivative can be found via $h'(x) = (1)\cdot(2x) = 2x$. However, this cannot be correct since $h(x) = x \cdot x^2$ simplifies to $h(x) = x^3$ so, in fact, $h'(x) = 3x^2$. Thus, in general, $\frac{d}{dx}\left[f(x)g(x)\right] \neq f'(x)g'(x)$. In words, the derivative of a product is **not** the product of its derivatives. Here is the theorem we are looking for.

Theorem (Product Rule): If $f(x)$ and $g(x)$ are differentiable functions, then $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$.

In words, this says that the derivative of a product is given by "first function times the derivative of the second plus the second times the derivative of the first." Because the rule is somewhat cumbersome, shorthand is nice: $\boxed{(uv)' = uv' + vu'}$.

Example: Find the derivative of $f(x) = (3x+4)(x^2 - 5x)$.

Solution: Use the Product Rule:

$$\begin{aligned} f'(x) &= (3x+4)\frac{d}{dx}(x^2 - 5x) + (x^2 - 5x)\frac{d}{dx}(3x+4) \\ &= (3x+4)(2x-5) + (x^2 - 5x)(3+0) \\ &= 6x^2 - 7x - 20 + 3x^2 - 15x \\ &= 9x^2 - 22x - 20. \end{aligned}$$

Notice that an alternative route can be taken. First, you could do the algebra and then follow it by applying the theorems learned earlier. That is,

$$f(x) = (3x+4)(x^2-5x) = 3x^3 - 11x^2 - 20x$$

so then $f'(x) = 9x^2 - 22x - 20$. This serves as a nice way to check your answer!

Finally, here is the Quotient Rule.

<u>Theorem (Quotient Rule)</u>: If $f(x)$ and $g(x)$ are differentiable functions, then
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}.$$

In words, this says that the derivative of a quotient is given by "bottom function times the derivative of the top minus the top times the derivative of the bottom all over the bottom squared." Again, because the rule is somewhat cumbersome, shorthand is nice:
$$\boxed{\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}}.$$

<u>Example</u>: Find the derivative of $f(x) = \dfrac{3x-1}{x^2+2}$.

<u>Solution</u>: Use the Quotient Rule:

$$\begin{aligned} f'(x) &= \frac{(x^2+2)\frac{d}{dx}(3x-1) - (3x-1)\frac{d}{dx}(x^2+2)}{(x^2+2)^2} \\ &= \frac{(x^2+2)(3) - (3x-1)(2x)}{(x^2+2)^2} \\ &= \frac{3x^2+6-6x^2+2x}{(x^2+2)^2} \\ &= \frac{-3x^2+2x+6}{(x^2+2)^2}. \end{aligned}$$

It is fairly obvious that the Quotient Rule is the most complex of all rules discussed so far.

(1) There are some nice connections between the Product Rule and Quotient Rule that we will talk about in class.
(2) We'll probably prove one of the rules in class.