MATH 166
Lesson 2.4
Derivatives of Trigonometric Functions

In this section, we develop rules for differentiating the six trig functions: sine, cosine, tangent, secant, cosecant, and cotangent. It shouldn't be surprising that the derivatives of sine and cosine pave the way for the remaining four.

We begin with an informal approach to finding $\frac{d}{dx}(\sin x)$. Similar to Section 2.2, we can look at the graph of $f(x) = \sin x$, sketch several tangent lines to the graph, approximate their slopes, and then use this information to *sketch* the derivative. Take a look at the graph below (**Fig 1**).

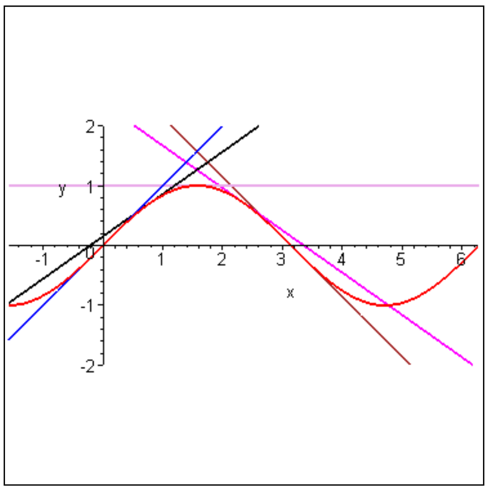


Fig 1. Sine with tangent lines

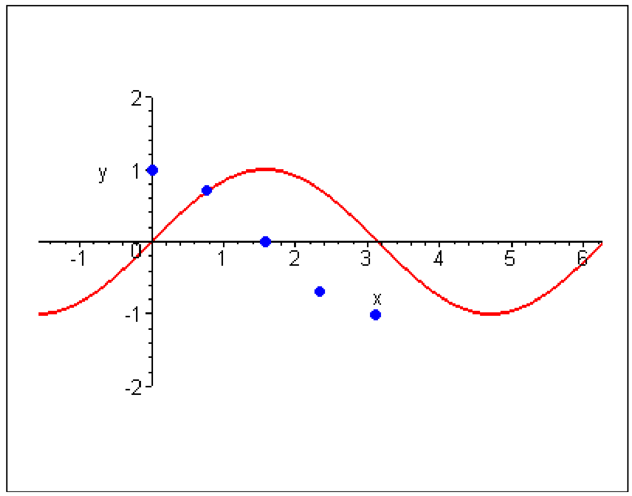
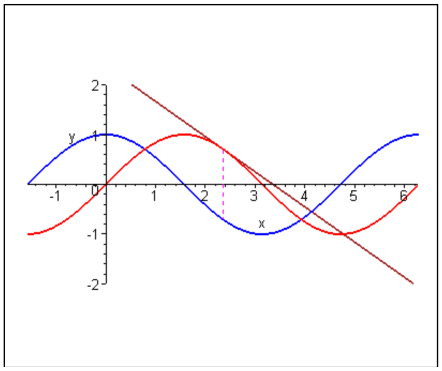


Fig 2. Plotting the slopes as outputs of a function

In addition to $f(x) = \sin x$, the tangent lines are sketched at $x = 0$, $\pi/4$, $\pi/2$, $3\pi/4$, and π. The corresponding slopes are roughly $m = 1,\ 0.7,\ 0,\ -0.7$, and -1. If we now plot the points $(0,1)$, $(\pi/4, 0.7)$, $(\pi/2, 0)$, $(3\pi/4, -0.7)$, and $(\pi, -1)$, these points lie on the graph of the **derivative** of $f(x) = \sin x$. Look at the points above (**Fig 2**). These points appear to lie on the graph of $y = \cos x$! We state this as a theorem.

<u>Theorem</u>: $\frac{d}{dx}(\sin x) = \cos x$. We'll prove it in class!

The theorem tells us that the <u>slope of the tangent line</u> to $f(x) = \sin x$ is given by <u>the value of the cosine graph</u> at that corresponding point. For example, when $x = 3\pi/4$, the tangent line to the sine graph has a slope of about $m = -0.7$. In different terms, this tells us that the cosine graph passes through the point $(3\pi/4, -0.7)$ (roughly). See the diagram below.



The next theorem shouldn't come as a great surprise.

<u>Theorem</u>: $\frac{d}{dx}(\cos x) = -\sin x$.

At this point, we are ready to find the derivatives of the other trig functions. For example, how would you find $\frac{d}{dx}(\tan x)$? Well, since the tangent function is defined as $\tan x = \frac{\sin x}{\cos x}$, you can use the Quotient Rule to find it. Here we go . . .

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{(\cos x)^2} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}$$

In an analogous fashion, you can find the derivatives of the other trigonometric functions using similar techniques. Furthermore, you can immediately apply the rules from the previous sections to new problems involving trigonometric functions (much of this we'll do in class). To wrap up, here are the derivatives of tangent, cotangent, secant, and cosecant.

<u>Theorem</u>: (a) $\frac{d}{dx}(\tan x) = \sec^2 x$ (b) $\frac{d}{dx}(\cot x) = -\csc^2 x$

(c) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (d) $\frac{d}{dx}(\csc x) = -\csc x \cot x$