MATH 166
Lesson 2.5a
The Chain Rule

In this section, we will study the most general derivative rule in the entire course. You will find the Chain Rule so important that many claim you will "seldom again differentiate any function **without** using it." First, look at the comparisons below:

Functions that we can currently differentiate	Functions we can differentiate with the Chain Rule
$y = x^2 + 3$	$y = \left(x^2 + 3\right)^5$
$f(x) = \cos x$	$f(x) = \left(\cos(6x)\right)^{3/2}$
$g(x) = \dfrac{x}{\sec x}$	$g(x) = \dfrac{x}{\sec x^2}$

You probably observe that the functions in the right-hand column are just more complicated versions of those seen in the left-hand column. After finishing this section, you'll be able to differentiate functions that appear formidable—even more complicated than the functions seen here. Before we begin, let's do a short review on **composition of functions**; this is crucial to understanding the Chain Rule.

Example: Given $f(x) = x^2 + 1$, $g(x) = \sqrt{x}$, and $h(x) = \sin x$, find
(a) $g\left(f(x)\right)$ and (b) $h\left(g\left(f(x)\right)\right)$.

Solution: (a) You can write $g\left(f(x)\right) = g\left(x^2+1\right) = \sqrt{x^2+1}$ so $g\left(f(x)\right) = \sqrt{x^2+1}$. It is a good idea to identify the "inner" and "outer" functions; the "inner" function is x^2+1 and the "outer" function is $\sqrt{x}$.

(b) Since we have already found $g\left(f(x)\right) = \sqrt{x^2+1}$ from part (a), we can say that $h\left(g\left(f(x)\right)\right) = h\left(\sqrt{x^2+1}\right) = \sin\sqrt{x^2+1}$. Here, the "inner" function is $\sqrt{x^2+1}$ and the "outer" function is $\sin x$. This problem is a bit trickier since the "inner" function $\sqrt{x^2+1}$ has—within itself—an "inside" and "outside" part.

Understanding this "decomposition" in the example above provides the foundation for grasping the Chain Rule. You need to become skilled in identifying the "inside" and "outside" functions.

Here is the general statement of the Chain Rule.

Chain Rule: Consider the function $y = f\left(g\left(x\right)\right)$. Rename the inside function u so that $u = g\left(x\right)$. Then the statement $y = f\left(g\left(x\right)\right)$ can be broken down into two separate statements; that is, $\begin{cases} y = f\left(u\right) \\ u = g\left(x\right) \end{cases}$. The Chain Rule tells us how to find $\frac{dy}{dx}$. Specifically, it says that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

A big note here: In words, the Chain Rule says that in order to differentiate a composite function, **first differentiate the outside function** and then **multiply this by the derivative of the inside function**.

Example: Differentiate the following functions: (a) $y = \left(x^3 + 3\right)^2$ (b) $y = \sqrt{4x^2 - \sin x}$

Solution: (a) First identify $u = x^3 + 3$ (the inside function). Then we may express $y = \left(x^3 + 3\right)^2$ as the two statements $\begin{cases} u = x^3 + 3 & \text{(INSIDE)} \\ y = u^2 & \text{(OUTSIDE)} \end{cases}$. The Chain Rule then says

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left(2u\right) \cdot \left(3x^2\right) \\ &= 2\left(x^3 + 3\right) \cdot 3x^2 \quad \leftarrow \text{ substitute } u = x^3 + 3 \text{ back in} \\ &= 6x^2\left(x^3 + 3\right). \quad \leftarrow \text{ answer in terms of } x \end{aligned}$$

So we can conclude that $y' = 6x^2\left(x^3 + 3\right)$. Do you see an easier way to do this problem?

(b) Similarly, identify $u = 4x^2 - \sin x$ and then write $\begin{cases} u = 4x^2 - \sin x & \text{(INSIDE)} \\ y = \sqrt{u} & \text{(OUTSIDE)} \end{cases}$. Then we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left(\frac{1}{2}u^{-1/2}\right) \cdot \left(8x - \cos x\right) \\ &= \frac{1}{2\sqrt{u}}\left(8x - \cos x\right) \\ &= \frac{8x - \cos x}{2\sqrt{4x^2 - \sin x}}. \quad \leftarrow \text{Bring answer back in terms of } x \end{aligned}$$

So we can state that $\frac{dy}{dx} = \frac{8x - \cos x}{2\sqrt{4x^2 - \sin x}}$.