MATH 166
Lesson 2.5b
The Chain Rule

Onward with the Chain Rule…there is an important lesson in the next example.

Example: Find the derivative of $y=\left(x^2+1\right)^{50}$.

Solution: Just like in earlier examples, identify $u=x^2+1$ (the inside function). Then we may express $y=\left(x^2+1\right)^{50}$ as the two statements $\begin{cases} u=x^2+1 & \text{(INSIDE)} \\ y=u^{50} & \text{(OUTSIDE)} \end{cases}$. The Chain Rule then says that

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\cdot\frac{du}{dx}\\ &=50u^{49}\cdot(2x)\\ &=50\left(x^2+1\right)^{49}(2x)\\ &=100x\left(x^2+1\right)^{49}.\end{aligned}$$

So we have $y'=100x\left(x^2+1\right)^{49}$. Rather than just ending here, there is something important on which to elaborate. If you look back at the steps above, one step is especially noteworthy: $\frac{dy}{dx}=50\left(x^2+1\right)^{49}(2x)$. If we treat x^2+1 as a **single expression**, then we can use the Power Rule as we know it and then multiply by the derivative of the inside function; see the illustration below.

$$y=\left(x^2+1\right)^{50} \quad\Rightarrow\quad \frac{dy}{dx}=\underbrace{50\left(x^2+1\right)^{49}}_{\text{Power Rule here}}\;\underbrace{(2x)}_{\text{derivative of the inside function}}$$

If we can recognize this, then no u's are necessary and we can find derivatives much more efficiently. Look at the illustration below.

$$y=\sqrt{8-3x^2} \quad\Rightarrow\quad y=\left(8-3x^2\right)^{1/2} \quad\Rightarrow\quad \frac{dy}{dx}=\underbrace{\frac{1}{2}\left(8-3x^2\right)^{-1/2}}_{\text{Power Rule here}}\;\underbrace{(-6x)}_{\text{derivative of the inside function}}$$

$$y=\left(\frac{\sin x}{x}\right)^4 \quad\Rightarrow\quad \frac{dy}{dx}=\underbrace{4\left(\frac{\sin x}{x}\right)^3}_{\text{Power Rule here}}\;\underbrace{\left(\frac{x\cos x-\sin x}{x^2}\right)}_{\text{derivative of the inside function via the Quotient Rule}}$$

This is not a process you have to adhere to right away but it makes for less writing and sometimes fewer headaches. Notice that the examples above are probably not in simplified form but it is surprising how quickly we can find y' (and then worry about the algebra later). We generalize all of this in the next statement.

General Power Rule (Chain Rule Version): Given $y = \left[u(x)\right]^n$, the derivative is given by $\frac{dy}{dx} = n\left[u(x)\right]^{n-1}\frac{du}{dx}$.

Example: Find the derivatives: (a) $y = x^3\sqrt{1-x^2}$ (b) $y = \sin\left(12x^2\right)$

Solution: (a) First notice that the function is in a product form so we can apply the Product Rule. So begin by writing $y' = x^3 \cdot \frac{d}{dx}\sqrt{1-x^2} + \sqrt{1-x^2} \cdot \frac{d}{dx}\left(x^3\right)$. Spend a minute to focus on $\frac{d}{dx}\sqrt{1-x^2} = \frac{d}{dx}\left(1-x^2\right)^{1/2}$. Here, we can use the Generalized Power Rule (Chain Rule) as follows.

$$\begin{aligned}\frac{d}{dx}\left(1-x^2\right)^{1/2} &= \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)\\ &= \frac{-2x}{2\left(1-x^2\right)^{1/2}}\\ &= \frac{-x}{\sqrt{1-x^2}}.\end{aligned}$$

So we have

$$\begin{aligned}y' &= x^3 \cdot \left(\frac{-x}{\sqrt{1-x^2}}\right) + \sqrt{1-x^2} \cdot \left(3x^2\right)\\ &= \frac{-x^4}{\sqrt{1-x^2}} + 3x^2\sqrt{1-x^2}\end{aligned}$$

This answer above is fine but you may get a common denominator to arrive at the more compact answer $y' = \frac{3x^2 - 4x^4}{\sqrt{1-x^2}}$.

(b) We now study $y = \sin\left(12x^2\right)$. Here, use $u = 12x^2$ along with $y = \sin u$. Then the Chain Rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u) \cdot (24x) = 24x\cos\left(12x^2\right).$$