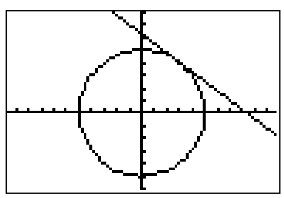
MATH 166
Lesson 2.6
Implicit Differentiation

Q: Why are we uneasy with finding the derivative of $\sin(x+y) = y^2 \cos x$?

A quick answer may be something along the lines of, "This doesn't look like other functions that we have differentiated such as $y = x^2 - 3x + 9$ or $y = \sqrt{x}\sin x$. In other words, y has not been isolated on one side of the equation. Something such as $\sin(x+y) = y^2 \cos x$ is defined **implicitly** (contrast this with **explicit form** $y = f(x)$). Naturally, if we could solve for y, we would do exactly this and then differentiate the function as usual. However, in situations such as the one above, it may be difficult (sometimes impossible) to do this. This section shows us how we can find the derivative of any function/relation without being restricted to the explicit form $y = f(x)$.

<u>Example</u>: Given $x^2 + y^2 = 25$, find $\frac{dy}{dx}$. Then find the equation of the tangent line to the graph at the point $(3,4)$.

<u>Solution</u>: Recall that the graph of $x^2 + y^2 = 25$ is a circle set at the origin with radius 5. This problem is asking us to find the equation of the tangent line to the circle at the point $(3,4)$. See the picture below.



In order to find $\frac{dy}{dx}$, let's first solve for y. $x^2 + y^2 = 25$ gives $y^2 = 25 - x^2$ so $y = \pm\sqrt{25 - x^2}$. Since we are looking at the point $(3,4)$, we can consider the *top half* of the circle. Thus, we can use the positive sign on the root so we have $y = \sqrt{25 - x^2}$. Now that we have explicit form, we can find the derivative. This involves the Generalized Power Rule (Chain Rule). So we have $y = \left(25 - x^2\right)^{1/2}$ which gives

$$y' = \frac{1}{2}\left(25 - x^2\right)^{-1/2}(-2x) = \frac{-2x}{2\left(25 - x^2\right)^{1/2}} = \boxed{\frac{-x}{\sqrt{25 - x^2}}}.$$

So then $\left.\frac{dy}{dx}\right|_{(3,4)} = \frac{-3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$. (This slope looks reasonably accurate when glancing at the line in the picture.) Then we can find the equation of the line by using the point-slope formula $y - y_1 = m(x - x_1)$ so we obtain $y - 4 = -\frac{3}{4}(x - 3)$. This will simplify to $y = -\frac{3}{4}x + \frac{25}{4}$. This is the equation of the tangent line seen in the graph.

Notice that the reason for our success above is the feasibility of first solving for y. Now we show an alternate route; **differentiating implicitly** by treating y as a function of *x*.

Example: Given $x^2 + y^2 = 25$, find $\frac{dy}{dx}$ by implicit differentiation. That is, treat $y = y(x)$.

Solution: To show how y depends on *x*, we can write the original equation as $x^2 + [y(x)]^2 = 25$. Now we'll differentiate both sides of the equation with respect to *x*: $\frac{d}{dx}\left(x^2 + [y(x)]^2\right) = \frac{d}{dx}(25)$. This gives $\frac{d}{dx}(x^2) + \frac{d}{dx}[y(x)]^2 = 0$. Now study this equation carefully. We know that $\frac{d}{dx}(x^2) = 2x$ but $\frac{d}{dx}[y(x)]^2$ warrants special attention because there is an "inside" function $y(x)$. For this part, use the Power Rule and Chain Rule to get $\frac{d}{dx}[y(x)]^2 = 2[y(x)]^1 y'(x)$. The expression $2[y(x)]^1 y'(x)$ is often abbreviated as $2yy'$. This means that $\frac{d}{dx}(x^2) + \frac{d}{dx}[y(x)]^2 = 0$ now reads $2x + 2yy' = 0$. Finally, solve for y' and simplify and you'll get $y' = -\frac{x}{y}$.

A couple of notes here:

1. Notice that the derivative $y' = -\frac{x}{y}$ contains both *x* and y. This is to be expected since we didn't solve for y in the first place.
2. Also check that the answer is the same as the one obtained earlier. Since we know that $y = \sqrt{25 - x^2}$, we can say that
$$y' = \underbrace{-\frac{x}{y}}_{\text{answer from second attempt}} = \underbrace{\frac{-x}{\sqrt{25 - x^2}}}_{\text{answer from first attempt}}.$$
The answers are equivalent.
3. To get the slope of the line, we use $y' = -\frac{x}{y}$ and we can say $y'\big|_{(3,4)} = -\frac{3}{4}$, agreeing again with the earlier example.