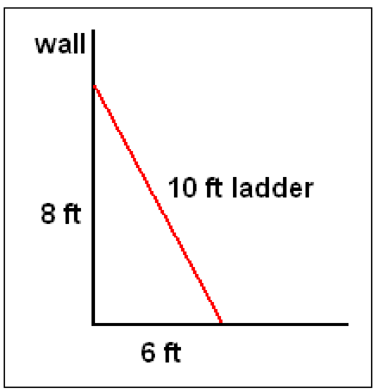
MATH 166
Lesson 2.8
Related Rates

We study dynamical problems in this section. This is to say that everything is dependent on *time*. As time goes by, the situation in these problems is most likely altered. The main tools that we will put to work here are (1) differentiation **with respect to time**, and (2) the **Chain Rule**. We start by looking at a problem.

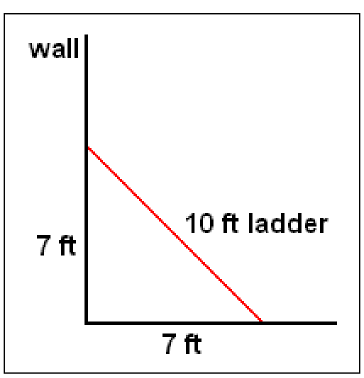
Example: A ladder 10 feet in length rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 foot/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

Discussion: A common misconception here is to say that, "The rate at which the ladder slides *down* the wall is always 1 foot/sec, the same rate at which the ladder is being pulled *away* from the wall." Why is this not accurate? Read on . . .

One possible scenario is seen below.



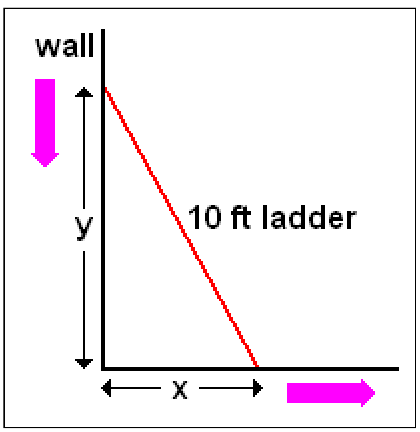
In the above picture, the ladder reaches 8 feet high on the wall and its base is 6 feet from the wall. What makes this possible is the Pythagorean Theorem: $10^2 = 6^2 + 8^2$. Now if we believe that tugging at the base at 1 foot/sec will result in the ladder sliding down 1 foot/sec, then—after 1 second passes—the picture will look like so (8 feet drops to 7 feet on the vertical and 6 feet grows to 7 feet on the horizontal):



The reason this isn't feasible is that $7^2 + 7^2 \neq 10^2$. The two rates can only be the same (in general) if either (1) the ladder is warping/bending in some way, or (2) the length of the ladder changes. Neither of these is happening so we can conclude that the rate at which one tugs at the base of the ladder is not necessarily the same as the rate at which the ladder slides down the wall. There could be a precise moment in time when they match, but in general, one of them will vary with time. Here are some guidelines that we will follow in the future to solve problems like this.

1. Draw a picture! Assign appropriate variables in the picture.
2. Identify the given information as well as information to be found.
3. Write down an equation that describes the situation (even as the situation changes).
4. Differentiate with respect to time.
5. Use the given information and solve for the unknown.

A useful picture here would be something like this:



The arrows are included to indicate the general motion of the ladder (*away* from the wall and *down* the wall). Notice we label the lengths as x and y because they are constantly changing in the problem; as the ladder moves, the x and y take on different values. It is not helpful to label the sides as 6 feet or 8 feet because the moment the ladder begins to *move*, these lengths are no longer correct!

Since the ladder slides away at a rate of 1 foot/sec, we can say $\frac{dx}{dt} = 1$ ft/sec. That is, the rate at which *x* changes with respect to time is 1 ft/sec (*x* is increasing). We seek the rate at which the ladder is falling when $x = 6$ ft. In other words, we want $\frac{dy}{dt}$ when $x = 6$ ft.

Bottom line here: **We are using the rate of change interpretation of the derivative to solve problems that involve movement and change.** The equation that links all pieces in this problem is $x^2 + y^2 = 10^2$. Notice that even as the ladder moves, this equation stays valid since *x* and *y* are free to take on many values. We'll finish this problem in class!!