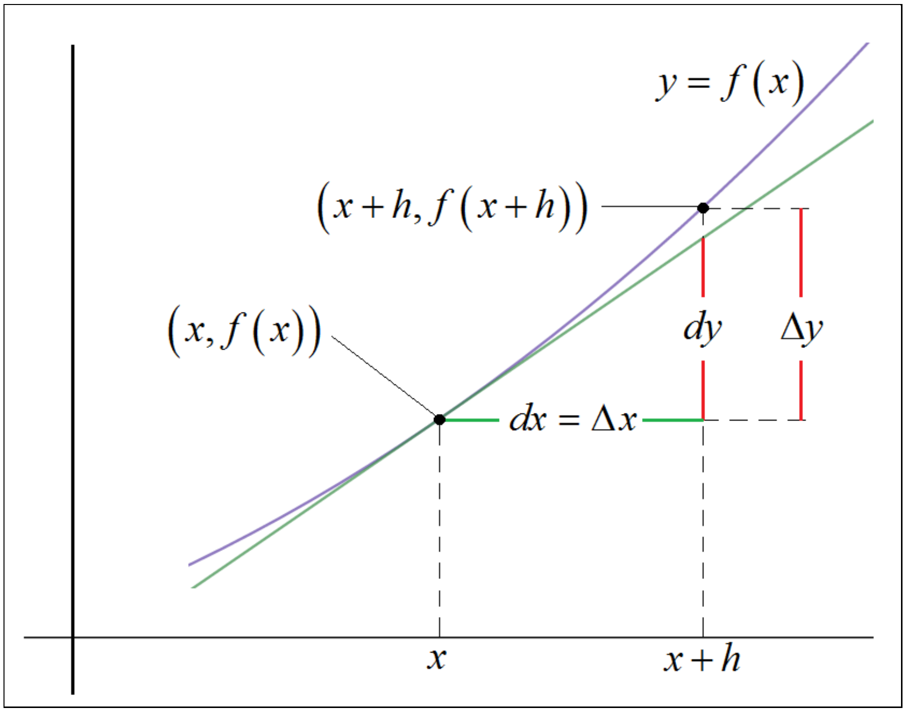
MATH 166
Lesson 2.9
Linear Approximations & Differentials

This lesson covers the concept of a *differential*. Differentials are useful when making any kind of approximation. Modern calculating tools use the idea of differentials when they spit out answers. We will begin by taking a look at a typical picture—one that has a curve with a tangent line at a point:



There is some complicated notation above. Let's talk about it.

$\Delta x = h = dx =$ change in x values (horizontal change) $\frac{dy}{dx} = f'(x) =$ the slope of the tangent line Then for the horizontal movement from x to $x+h$, we have $\Delta y =$ actual change in the y values of the <u>function</u> (vertical change) $dy =$ change in y values for the <u>tangent line</u>

For the first time in this class, we are giving concrete meaning to the symbols dx and dy (together which make $\frac{dy}{dx}$, the slope of the tangent line). We call dx the differential of x and dy the differential of y. It should come as no surprise that $dx = \Delta x$. The big thing to notice in the picture is that . . .

For Δx or dx small, $\Delta y \approx dy$. In plain terms, if the horizontal change is small, then the change in the function values can be estimated by the change in the tangent line. They are nearly equal. Notice that the smaller the Δx, the closer the Δy and dy values.

Since $\frac{dy}{dx} = f'(x)$, we can write $dy = f'(x)dx$. Using the above statements, we can say that $\Delta y \approx f'(x)dx$ as long as Δx is small. Here is an example that should clarify things.

Example: Let $f(x) = x^3$.

(a) Find dy when $x = 1$ and $dx = 0.1$.
(b) Find Δy when $x = 1$ and $\Delta x = 0.1$. Discuss.

Solution: (a) Use $dy = f'(x)dx$ so that $dy = 3x^2 dx$. Then $dy = 3(1)^2(0.1) = \boxed{0.3}$.
(b) Since Δy is the actual change in the function, think of the new function value minus the old; that is, $\Delta y = f(x + \Delta x) - f(x)$. So $\Delta y = f(1 + 0.1) - f(1) = (1.1)^3 - 1^3 = \boxed{0.331}$.
The message to take from this is that dy and Δy are very close. Notice that part (a) involves less work.

The next concept will borrow an idea from the previous problem. Given that $\Delta y = f(x + \Delta x) - f(x)$, we can rearrange this to give $f(x + \Delta x) = \Delta y + f(x)$. Using the fact that $\Delta y \approx dy$, we can state that $f(x + \Delta x) \approx dy + f(x)$ or that

$$\boxed{f(x + \Delta x) \approx f'(x)dx + f(x)}.$$

This is useful in the following way. Suppose we wished to approximate the value of $\sqrt{9.3}$. The main function here is the square root so let $f(x) = \sqrt{x}$. We will pick $x = 9$ since the square root of 9 is an easy calculation. Because we desire $\sqrt{9.3}$, choose $\Delta x = dx = 0.3$. Now we use the formula $f(x + \Delta x) \approx f'(x)dx + f(x)$ to get $\sqrt{x + \Delta x} \approx \frac{1}{2\sqrt{x}}dx + \sqrt{x}$. Put $x = 9$ and $\Delta x = dx = 0.3$ to get

$$\sqrt{9.3} \approx \frac{1}{2\sqrt{9}}(0.3) + \sqrt{9}.$$

Simplifying the right-hand side, we get $\sqrt{9.3} \approx 3.05$. This bears with intuition since we would predict that $\sqrt{9.3} > \sqrt{9}$. You may wish to check the value of $\sqrt{9.3}$ on a hand held calculator. For example, the TI-84 gives 3.04959…. Our 3.05 is a great approximation.