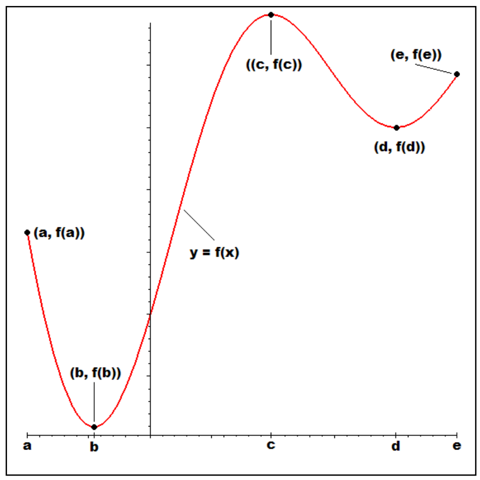
MATH 166
Lesson 3.1
Maximum and Minimum Values

In this lesson (and entire unit), we will be using the derivative $f'(x)$ to provide information about the graph $y = f(x)$. We'll begin with some terminology. Look at the graph below.

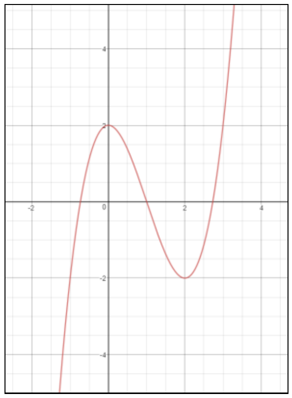


The low points and high points of the graph are called **extreme values**. Some people just use the words **minimum** and **maximum**. The maximum value of the graph is $f(c)$; the minimum value is $f(b)$. Notice that maximum/minimum values are *output values* of the function (that is, y values). It is also acceptable to say that the maximum *occurs* at $x = c$ while the minimum *occurs* at $x = b$. Here is a very important theorem.

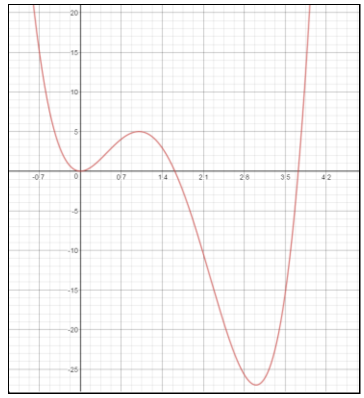
<u>Extreme Value Theorem</u>: If $f(x)$ is continuous on a <u>closed</u> interval, then $f(x)$ has both a minimum and maximum value.

The picture above confirms this. There we have a continuous function with the points $(a, f(a))$ and $(e, f(e))$ included—so the interval is closed since the endpoints are included. Note that the graph has both a minimum and a maximum as cited above.

Here is another bit of terminology. The **local maximum** and **local minimum** are the "hills" and "valleys" that occur within a specified interval. In other words, a local maximum may be the highest point on a function if you stay in and around the maximum; however, it need not be the highest point of the function overall (this would be called an **absolute maximum**). A similar statement holds for a local minimum and an **absolute minimum**. We'll look at examples to clarify this.



$f(x) = x^3 - 3x^2 + 2$

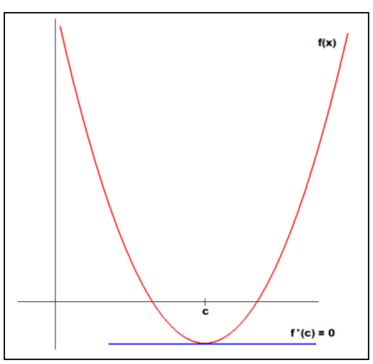


$g(x) = 3x^4 - 16x^3 + 18x^2$

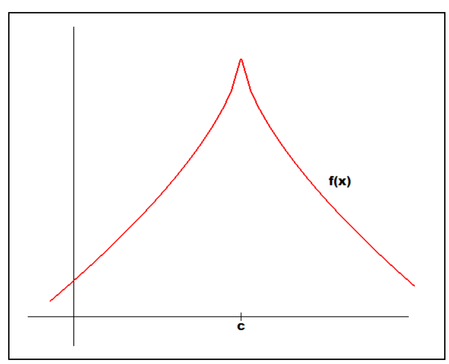
Notice that the graph of $y = f(x)$ has a peak at $(0,2)$ and a valley at $(2,-2)$; these correspond to a local maximum and local minimum, respectively. They are not "absolute" because there are higher and lower points—the graph takes off to infinity and negative infinity. As for $y = g(x)$, there is a local minimum at $(0,0)$ and a local maximum at $(1,5)$. Notice that $(3,-27)$ is an absolute minimum since it is the lowest point on the graph. We now state the most important definition of this section. This sets the stage for everything else in this lesson.

<u>Definition</u>: Let $f(c)$ be defined. If $f'(c) = 0$ or $f'(c)$ is undefined, then *c* is called a **critical number** of f.

Why do you suppose we call such points **critical numbers**? Let's look at the two scenarios:



$f'(c) = 0$



$f'(c)$ is undefined

What do both of the pictures above have in common? (Answer: The critical numbers lead to extreme values.) In summary, we can find the minimum and maximum values of functions by first (a) finding the derivative, (b) finding critical numbers, and (c) testing to see where we get the largest number (maximum) and smallest number (minimum).