MATH 166
Lesson 3.2
Mean Value Theorem

The main topic in this lesson is the Mean Value Theorem for derivatives; the math people of this world would argue that this is one of the most important theorems in Calculus. It represents the thread that ties many of the ideas together. Here it is:

Mean Value Theorem: Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a value *c* in (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

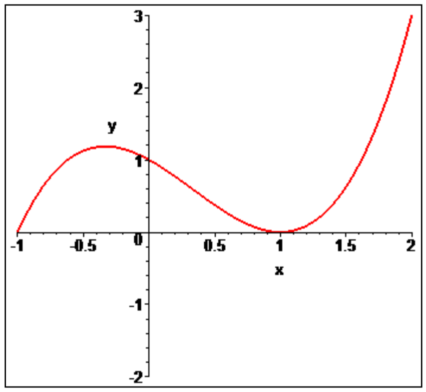
A few notes are in order here.

1. This theorem is regarded as an "existence" theorem. It guarantees the existence of a value *c* but it in no way tells you *how* to get it. For some problems, finding *c* is a cinch; in others, it may be nearly impossible. In the latter case, we'll just use a calculator to estimate it.
2. The theorem uses the language "there exists a value *c* in (a,b) such that . . ." It does not state *how many c* values there are. In all cases, you'll be able to find at least one—in some cases, two, three, or more.
3. Note that the word "mean" refers to "average." See note 5 below.
4. The subtleties of the closed interval $[a,b]$ and the open interval (a,b) are not terribly important for what we'll do in this class. Just remember that the assumptions of continuity and differentiability (smoothness) must be met.
5. Probably the most reasonable thing to ask is what this theorem states from a *geometric* perspective. Recall that $f'(c)$ gives the **slope of the tangent line** (instantaneous rate of change) at $x=c$. On the other hand, $\frac{f(b)-f(a)}{b-a}$ is just an ordinary slope; it is the change in *y* over the change in *x* using the endpoints of the interval $[a,b]$. That is, it represents the **slope of the secant line** (average rate of change) through the endpoints. Probably the most important thing here is the "equals" sign between the two. In words, the Mean Value Theorem (MVT) tells us that there is a location *c* in the interval (a,b) such that **the slope of the tangent line is equal to the slope of the secant line**. See the breakdown below.

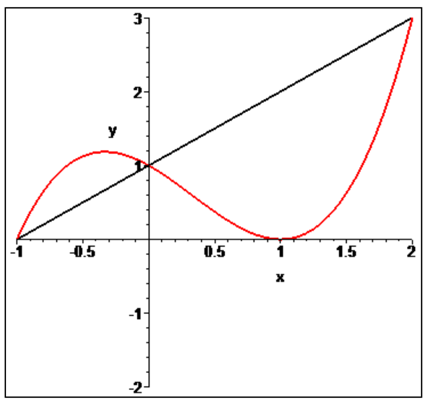
Geometric Interpretation:

$$\underbrace{f'(c)}_{\text{slope of the tangent line}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of the secant line through the endpoints}}$$

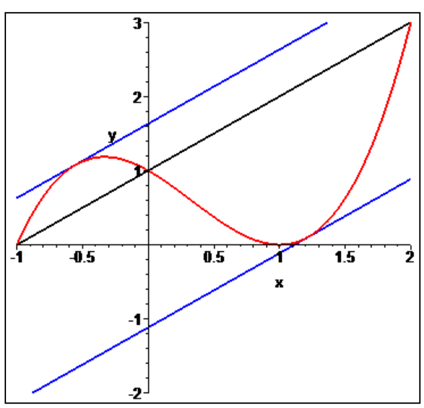
Now we all know that a picture is worth a thousand words. Consider a smooth and continuous function $y = f(x)$ on the window $[-1, 2]$:



Now draw the secant line passing through the endpoints of the interval:



Then ask, are there any tangent line(s) in the interval $(-1, 2)$ with the **same slope** as the secant line slope? There appears to be two . . .



By the naked eye, the *c* values are around $c \approx -0.5$ and $c \approx 1.25$. This is what the MVT is all about!