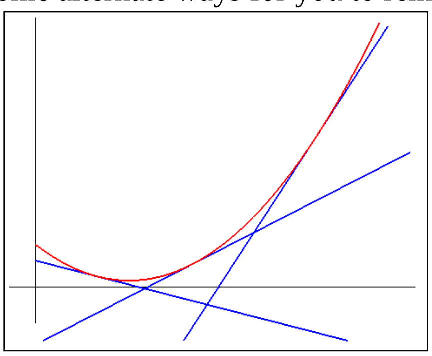
MATH 166
Lesson 3.3b
Concavity and Points of Inflection

This section is mostly about looking at f'' (the second derivative) and seeing what kind of information we can gather about the graph of $f(x)$. This first characteristic is called **concavity**.

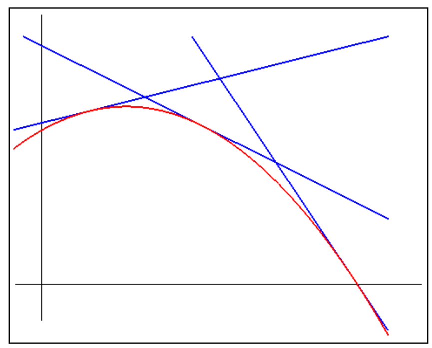
Definition: Let $f(x)$ be differentiable on an interval.

1. The graph of $f(x)$ is **concave upward** if f' is increasing.
2. The graph of $f(x)$ is **concave downward** if f' is decreasing.

Some alternate ways for you to remember this are below:



CONCAVE UPWARD
f' is increasing
Graph of $f(x)$ is ABOVE the tangent lines



CONCAVE DOWNWARD
f' is decreasing
Graph of $f(x)$ is BELOW the tangent lines

A graph that is concave upward would hold water if you were to dump water on the graph. On the contrary, the water would run off a graph if it were concave downward. For a graph that is concave upward, do you see why it is appropriate to say that f' is increasing? If you look carefully at the tangent lines in the picture, the first one has negative slope, the second one has positive slope, and the last one has a positive slope (even steeper). Thus, the slopes get larger from left to right; hence, f' is increasing. Convince yourself of the analogous result when $f(x)$ is concave downward.

The next result is the test for concavity. Since we want to know where f' is increasing and decreasing, we can look at the sign of f'' (the second derivative) to give us this information. Similar to our increasing/decreasing game from before, the sign of $f''(x)$ tells all about the concavity of the graph.

Test for Concavity

1. If $f''(x) > 0$ on an interval, then the graph of $f(x)$ is concave upward there.

2. If $f''(x) < 0$ on an interval, then the graph of $f(x)$ is <u>concave downward</u> there.

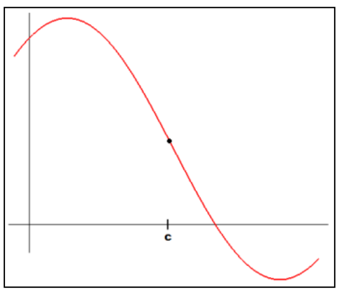
The last idea in this section is that of an **inflection point**. Here is the definition.

<u>Definition</u>: The point $(c, f(c))$ is a **point of inflection** to the graph of $f(x)$ if

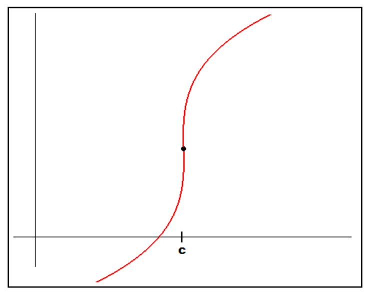
1. the concavity changes at $x = c$, and
2. the tangent line exists at $x = c$ (or the graph is smooth at $x = c$).

Let's look at some pictures.

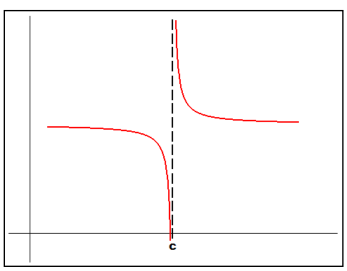
PICTURE A:



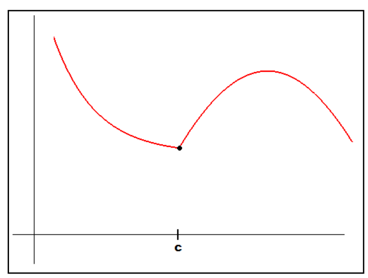
PICTURE B:



PICTURE C:



PICTURE D:



In reviewing the pictures above, all graphs have a change in concavity at $x = c$ so the first condition is easily met. Now let's check the second condition. For PICTURE A, the tangent line exists at $x = c$ so this means that $(c, f(c))$ is an inflection point. For PICTURE B, the tangent line exists at $x = c$ so this means that $(c, f(c))$ is an inflection point. Notice that the *slope* of the tangent line fails to exist—it is infinite. However, you can sketch the tangent line; there will be a vertical tangent at $x = c$. For PICTURE C, the tangent line fails to exist at $x = c$. Also notice that $(c, f(c))$ is not even a point on the graph; there appears to be a vertical asymptote at $x = c$ so no inflection point here. Lastly, the graph in PICTURE D shows a sharp change in direction at $(c, f(c))$ so the graph is not smooth there. We conclude that $(c, f(c))$ is not a point of inflection.
In summary, although all four graphs have a change in concavity at $x = c$, <u>only PICTURES A and B have points of inflection</u>.