MATH 166
Lesson 3.3c
Second Derivative Test

This lesson continues to examine how to find extreme values but does so by proposing a test different from the First Derivative Test. Here, we find extreme values by using the *second* derivative. Naturally, it is called the Second Derivative Test.

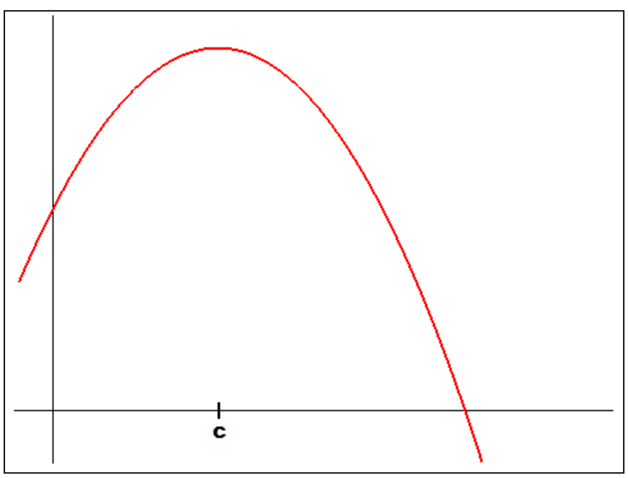
<u>Second Derivative Test</u>

Let $f'(c)=0$ (so $x=c$ is a critical number of $f(x)$).

1. If $f''(c)>0$ then $f(c)$ is a relative minimum.
2. If $f''(c)<0$ then $f(c)$ is a relative maximum.
3. If $f''(c)=0$ then the test is inconclusive. Use the First Derivative Test.

<u>Note</u>: In the event of the third case above, don't panic. This just means you have to go back to analyzing the sign of $f'(x)$ to determine the nature of the extreme value.

Let's take a look at why this test makes sense. For example, statement 2 says "If $f''(c)<0$ then $f(c)$ is a relative maximum." We know that $x=c$ is a critical number and the graph of $f(x)$ is concave downward in the vicinity of the critical number. Recall that a **negative second derivative** implies **downward concavity** (Lesson 3.3b). So the picture is reminiscent of:



The graph definitely has a maximum at $(c, f(c))$ as the test states.

<u>Note</u>: There is one main advantage to using the Second Derivative Test. In general, you can avoid making a sign chart for $f'(x)$. The downside is that some people confuse the

First and Second Derivative Tests. In words, the Second Derivative Test says that you must first find the critical numbers of the function and then test them in the **second derivative**. The sign of the result reveals what kind of extreme value you have. In short,

you insert numbers from the first derivative into the second derivative to examine concavity;

the sign tells you whether you have a minimum or maximum.

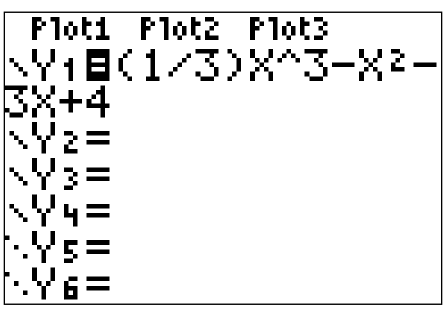
Here is an example that illustrates how the test works.

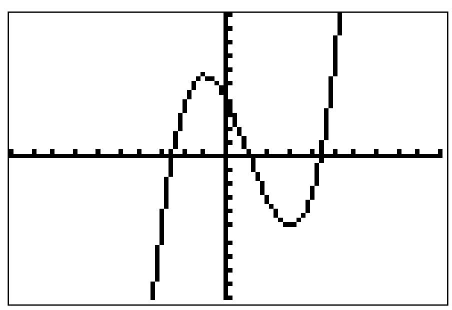
Example: Find the relative extrema of $y = \frac{1}{3}x^3 - x^2 - 3x + 4$ by employing the Second Derivative Test.

Solution: Find $y' = x^2 - 2x - 3 = (x-3)(x+1)$. Critical numbers come from $(x-3)(x+1) = 0$ so we get $x = -1$ and $x = 3$. *Hold on to this thought*. Now find $y'' = 2x - 2$. At this point, simply check the critical numbers in y''. We get

$$y''(-1) = 2(-1) - 2 = -4 < 0 \quad \text{(downward concavity)}$$
$$y''(3) = 2(3) - 2 = 4 > 0 \quad \text{(upward concavity)}$$

By the Second Derivative Test, there is a maximum at $x = -1$ and a minimum at $x = 3$. Get the *y*-coordinate of these extreme values by using $y = \frac{1}{3}x^3 - x^2 - 3x + 4$. The graphing calculator, as always, is a good way to confirm all of this:





Note: The Second Derivative Test can sometimes be "quicker" than the First Derivative Test. Find the critical numbers and then test them in the second derivative. It's short and sweet!