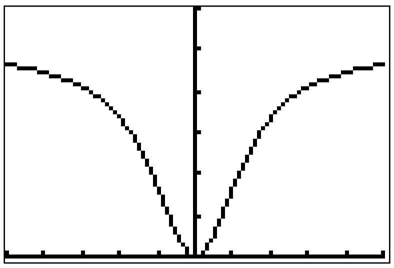
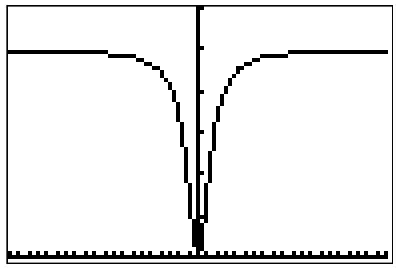
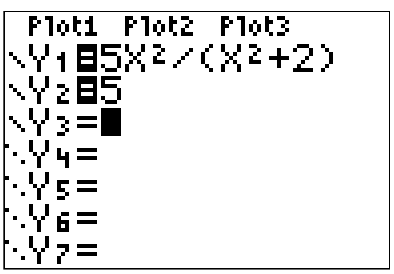
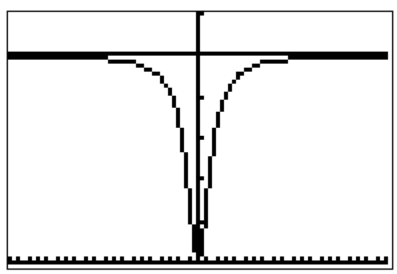
MATH 166
Lesson 3.4
Limits at Infinity

To say that we will study "limits at infinity" means that we will analyze expressions such as $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. For now, let's study the function $f(x)=\dfrac{5x^2}{x^2+2}$. Plot it on the calculator using the window $[-5,5,0,6]$ and then $[-20,20,0,6]$.

It seems that as x moves further and further away from the origin, $f(x)$ "levels out" close to 5. In symbols, $\lim_{x\to\infty} f(x)=5$ and $\lim_{x\to-\infty} f(x)=5$. You may recall from algebra that this leads to a **horizontal asymptote** at $y=5$. For example, if you plot the function $y=5$ with $f(x)=\dfrac{5x^2}{x^2+2}$, you can see how close the graph of $f(x)$ comes to the asymptote:





You may recall from algebra that

$$\lim_{x\to\infty}\frac{n(x)}{d(x)}=\boxed{\begin{array}{c}\text{ratio of the highest}\\ \text{degree coefficients}\end{array}}$$

if the functions $n(x)$ and $d(x)$ have the same degree. For the example above, since both the numerator and denominator of $f(x)=\dfrac{5x^2}{x^2+2}$ have degree 2, we could immediately conclude that $\lim_{x\to\infty}\frac{5x^2}{x^2+2}=\frac{5}{1}=5$. This is just a helpful hint to keep in mind. Here is a general statement.

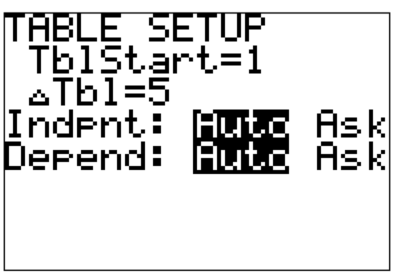
The line $y=L$ is a **horizontal asymptote** to the graph of $f(x)$ if either $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$.

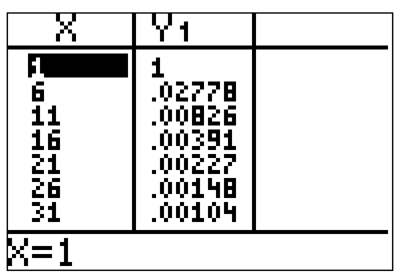
Here are some additional special limits:

For $r>0$ and c any constant, 1. $\lim_{x\to\infty}\frac{c}{x^r}=0$ and 2. $\lim_{x\to-\infty}\frac{c}{x^r}=0$

Example: Find $\lim_{x\to\infty}\frac{1}{x^2}$.

Solution: This is a direct application of $\lim_{x\to\infty}\frac{c}{x^r}=0$ as it appears above so you may write $\lim_{x\to\infty}\frac{1}{x^2}=0$. Intuitively, this should make sense since, as $x\to\infty$, $x^2\to\infty$. Then its reciprocal $\frac{1}{x^2}$ shrinks rapidly. This is why $\lim_{x\to\infty}\frac{1}{x^2}=0$. To check this on a calculator, you can define the function $y=\frac{1}{x^2}$ and look at the table for *very large x*.





As *x* grows, *y* shrinks to zero.

Finally, we close with an important result. We'll see an illustration of each case in class.

Limits at Infinity for Rational Functions

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit is given by the ratio of the highest degree coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit does not exist.