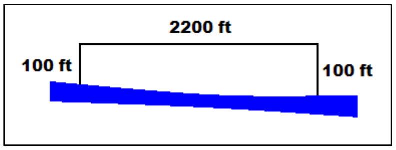
MATH 166
Lesson 3.7
Optimization

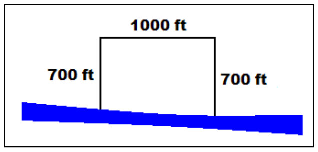
Question: A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field (length and width) with the largest area?

This is a typical scenario of "maximizing one's resources." Let's look at a few possibilities. Notice that all of the scenarios use the entire 2400 feet of fencing. Note: Pictures are not drawn to scale.

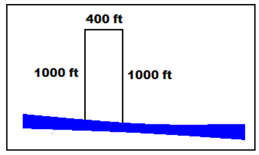
SCENARIO A:



SCENARIO B:

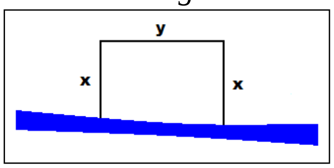


SCENARIO C:



In Scenario A, $\text{Area} = (2200 \text{ ft})(100 \text{ ft}) = 220,000 \text{ ft}^2$. In Scenario B, $\text{Area} = (1000 \text{ ft})(700 \text{ ft}) = 700,000 \text{ ft}^2$, more than three times the area from Scenario A! Finally, Scenario C gives $\text{Area} = (400 \text{ ft})(1000 \text{ ft}) = 400,000 \text{ ft}^2$. Of the three scenarios, SCENARIO B is clearly the best since we obtain the largest area. But is this the best of all possible arrangements? *This we do not know*. Looking at additional scenarios is not an efficient way to solve this problem since there are an infinite number of scenarios possible. This calls for doing a little bit of "modeling" with algebra.

Solution Path: Consider the *general* scenario below:



Unlike the other scenarios, we are not labeling the sides with fixed lengths. Rather, we give them variable names (x and y) with the goal of finding the x and y that **maximize** the area. So we write $A = xy$ with the intent of maximizing A (area). The problem with trying to find a maximum of this function is that we have **two** variables on the right-hand side of the equation (both an x and a y). But since we know that the farmer has 2400 feet of fencing, this translates to $2x + y = 2400$ (look at the diagram). If we solve this equation for y we get $y = 2400 - 2x$. Next, we can substitute this into our area equation. We get $A = xy = x(2400 - 2x) = 2400x - 2x^2$. The area equation now has **one** variable on the right-hand side so it is appropriate to use the functional notation $\boxed{A(x) = 2400x - 2x^2}$. **To maximize a function like this makes perfect sense; this is**

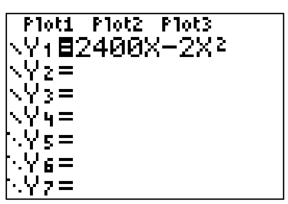
what we have been doing for the last three lessons. However, before diving into the Calculus, it is important to look at only the values that make sense to this problem. Since the main variable is *x*—we must ask—are there any restrictions on *x*? First, $x > 0$ since it represents the length of a side. Also, notice that $x < 1200$. Look back at the diagram. If $x = 1200$ feet, then the 2400 feet of fence would be used up on the sides and there would be nothing left for the portion parallel to the river. Thus, the upper bound on *x* is 1200 ft. Our problem becomes:

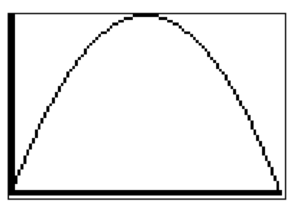
Maximize the function $A(x) = 2400x - 2x^2$ for *x* in the interval $(0,1200)$.

Notice the resemblance this has to problems from Lesson 3.1. However, it takes a lot of work to get to this stage! To maximize $A(x)$, find any critical numbers and test them. We get $A'(x) = 2400 - 4x$ so $2400 - 4x = 0$ leads to $x = 600$ (our critical number). We should test this value in $A(x)$ along with the endpoints $x = 0$ and $x = 1200$. Here is a summary in table form:

x	A(x)
0	0
600	720000
1200	0

It appears that the endpoints give *minimum* areas so we choose $x = 600$ feet since this gives us the largest area (720,000 ft^2). Notice how close we came to this value in SCENARIO B. There are two ways to verify that we, in fact, have found the largest area possible. We could use one of the tests from the previous sections (First Derivative Test or Second Derivative Test) or we could graph the function and make a conclusion from the graph. If we use the Second Derivative Test, notice that $A''(x) = -4 < 0$. Downward concavity implies a maximum value so all is well here. Another check is by graphing. Use the window $[0,1200,0,720000]$. Notice that the *x*'s use the domain $(0,1200)$ while the *y*'s cover the area values in the table from 0 to 720,000.





You can definitely see the maximum in the middle of the screen so $x = 600$ feet is the answer. Notice that even a slight deviation from this will result in a smaller area (e.g., $x = 550$ or $x = 630$ moves you away from the peak in the graph). Going back to $y = 2400 - 2x$, we find $y = 2400 - 2(600) = 1200$. So the dimensions we seek are $x = 600$ feet and $y = 1200$ feet and this results in a maximum area of 720,000 ft^2.