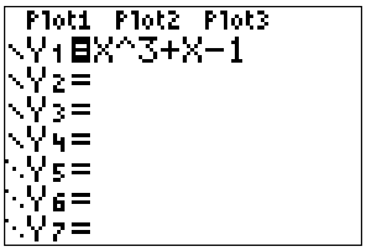
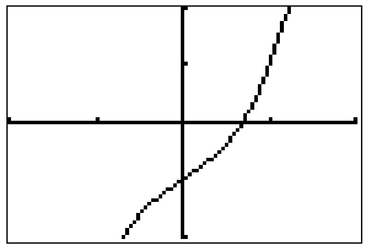
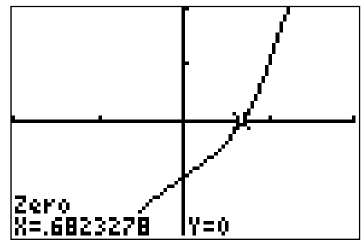
MATH 166
Lesson 3.8
Newton's Method

Consider finding an *x*-intercept of the function $f(x) = x^3 + x - 1$. This would mean setting $y = 0$ and solving the equation $x^3 + x - 1 = 0$, a challenging task! As an alternative, we could use the calculator's ZERO feature to "trap" the *x*-intercept and to nail down a good approximation. Here is some activity in the window $[-2, 2, -2, 2]$:

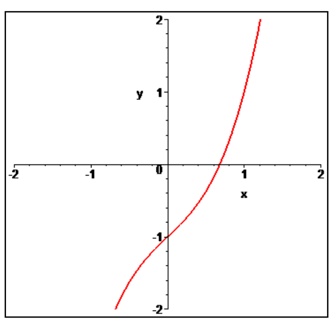




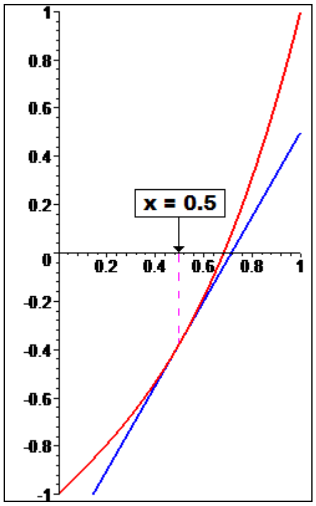


The calculator estimates the *x*-intercept to be around $x \approx 0.682$. <u>Have you ever wondered *how* the calculator is able to do such a thing</u>? This lesson will illustrate some of the Calculus behind this problem. Although finding an *x*-intercept seems mostly to be an algebra problem, it is the *Calculus* that allows us to take a difficult problem such as $x^3 + x - 1 = 0$ and reduce it to mere mechanics. Newton's method—the topic of this lesson—gives an algorithm for finding zeros (*x*-intercepts) of functions (among other things). We will use this opening example as motivation for the method.

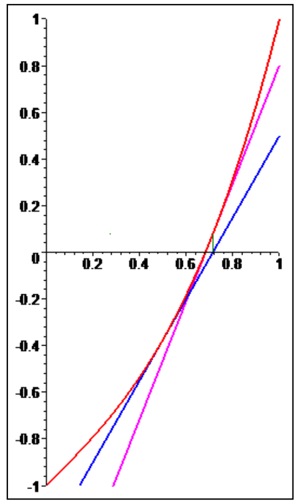
Again, here is the graph of $f(x) = x^3 + x - 1$ on the $[-2, 2, -2, 2]$ window.



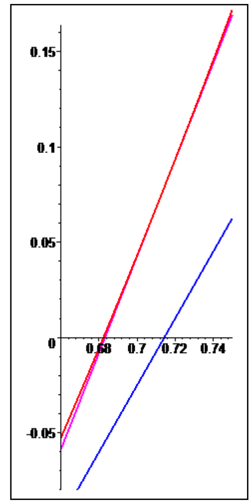
If we were to make a guess at the *x*-intercept, we might say (roughly) something like $x = 0.5$. Look what happens when we sketch the tangent line to the curve at $x = 0.5$. See Picture A, next page.



Picture A



Picture B



Picture B (zoomed in)

Notice that the tangent line appears to cross the *x*-axis at a location that is **closer** to the intercept we are seeking (Picture A). Thus, we'll use this new location (around $x = 0.7$) to calculate another tangent line there. Look at Picture B. It almost appears that the new tangent line (Picture B) has the same *x*-intercept as the cubic function. So just after finding two tangent lines, we seem to have a good approximation for the intercept to the original function. This technique can be summarized as follows:

1. Make an initial guess for the *x*-intercept. This can come from looking at the graph of the function.
2. Find the tangent line at the location of this initial guess.
3. Find the *x*-intercept of the tangent line found above.
4. Using the *x*-intercept from step 3, calculate a new tangent line there.
5. Repeat steps 3 and 4 until the values give a satisfactory approximation. The fact that we repeat this process means this is an *iterative scheme* and amenable to writing a small program to execute this procedure.

Below we state this algorithm; it is called Newton's Method for approximating zeros.

Newton's Method

Consider finding an *x*-intercept of the function $y = f(x)$ (or solving the equation $f(x) = 0$). First, make an initial guess; call it x_1. Then use the algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

with $n = 1, 2, 3, \ldots$ to obtain a sequence of values $x_1, x_2, x_3, \ldots$. Continue this until a noticeable repetition is found. ***We will derive Eq (1) in class and see an example.***