MATH 166
Lesson 3.9
Antiderivatives

We have been discussing differentiation and its applications for two chapters now. In this final section, we begin talking about the inverse operation—**antidifferentiation**. Much like *subtraction* is the opposite of *addition* and *division* is the opposite of *multiplication*, we will see *antiderivatives* as the opposite of *derivatives*. We open with a straightforward example.

Example: Find a function $F(x)$ whose derivative is $f(x) = 4x^3$.

Solution: If we know our differentiation rules well, we would say that $F(x) = x^4$. This is because $F'(x) = 4x^3 = f(x)$.

Based on the example above, we call $F(x)$ an **antiderivative** of $f(x)$. Why not *the* antiderivative? The simple answer is that there are many more correct antiderivatives. For example, we could have said $F(x) = x^4 + 3$ or $F(x) = x^4 - 5$ or $F(x) = x^4 + \pi$, etc. The list is endless! Because of this, the best answer is $F(x) = x^4 + C$, where C is any constant. Once we come up with $F(x) = x^4 + C$, it is clear that $F'(x) = f(x)$.

Note 1: It is common to denote an antiderivative by using the same lettered function but in CAPITAL style. For example, given a function $h(x)$, we'll typically call its antiderivative $H(x)$.

Note 2: Something to notice about the above problem is the inverse nature of the task. Rather than asking, "What is the derivative of this function?" we are asking, "From what function would you get this derivative?" The first question is along the lines of, "Here is the question. What is the answer?" On the other hand, the second question asks, "Here is the answer. What is the question?" From an application point of view, a scientist may know how fast a particle is moving but may wish to know the particle's position. Similarly, we may know the rate at which a faucet is leaking but want to know how much water has leaked after a certain amount of time. Each of these scenarios provides a *derivative* but wishes to know something about the *original function*.

Example: Given $f'(x) = x\sqrt{x}$ and $f(4) = 3$, determine the function $f(x)$.

Solution: Notice that antidifferentiation will give us $f(x)$ but with some degree of uncertainty (the arbitrary constant). This is where having the additional information $f(4) = 3$ should come in handy. So $f'(x) = x\sqrt{x} = x^{3/2}$. In order to find an antiderivative, we need to do literally the opposite of what we do for differentiation. For derivatives of powers, we multiply by the power and then drop the power by 1 (x^n becomes nx^{n-1}). For antiderivatives, we have to do the opposite and in the opposite order (x^n becomes $\frac{x^{n+1}}{n+1}$ because we increase the power by one and then divide by this power). Therefore, $f'(x) = x^{3/2}$ becomes $f(x) = \frac{x^{5/2}}{5/2} + C = \frac{2}{5}x^{5/2} + C$. Then we apply $f(4) = 3$ to get $f(4) = \frac{2}{5}4^{5/2} + C = 3$ so you can determine that $C = -\frac{49}{5}$. In other words, our sought after function is $\boxed{f(x) = \frac{2}{5}x^{5/2} - \frac{49}{5}}$. Note that a quick check of this answer should confirm that both $f'(x) = x\sqrt{x}$ and $f(4) = 3$, as given in the original statement of the problem.

We close with an **important** application.

<u>Example</u>: Suppose you toss a ball upward with a speed of 48 feet/sec from the edge of a cliff that is 432 feet above ground. How much time passes before the ball hits the ground?

<u>Solution</u>: We begin with $a(t) = -32$ (acceleration) where the units are ft/sec^2. We take the positive direction to be upward. Antidifferentiation gives $v(t) = -32t + C$ (velocity). However, we know that $v(0) = 48$ so we can figure out the value of C from this:

$$v(0) = -32(0) + C = 48,$$

so $C = 48$. Therefore, we can update the velocity function to read $v(t) = -32t + 48$. Another antiderivative will give $s(t) = -32\frac{t^2}{2} + 48t + D = -16t^2 + 48t + D$ (position). Here, we know the initial height is 432 feet so we apply $s(0) = 432$ to the preceding equation:

$$s(0) = -16(0)^2 + 48(0) + D = 432$$

so $D = 432$. At this point, we now have the position function (describing where the ball is) for any moment during its flight: $s(t) = -16t^2 + 48t + 432$. So when does the ball hit the ground? This translates to when $s(t) = 0$ so $0 = -16t^2 + 48t + 432$. You can solve this equation in a variety of ways to get $t \approx 6.9$ seconds (we ignore the negative value of t).