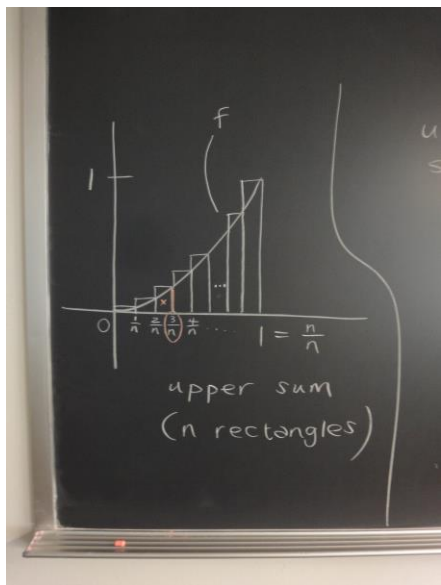


The Definite Integral

We played a little bit with the summation formulas & then did one BIG problem. Find the area under the curve $f(x) = x^2$ and above the x -axis from $x = 0$ to $x = 1$. See the four pictures below.

Picture 1:



Picture 2:

$$\begin{aligned} \text{upper sum} &= U(n) \\ &= \sum_{i=1}^n \underbrace{f(x_i)}_H \underbrace{\Delta x}_W \\ \text{width} &= \frac{1-0}{n} = \left(\frac{1}{n}\right) \\ \text{heights used} &: f\left(\frac{1}{n}\right), f\left(\frac{2}{n}\right), \dots, f(1) \\ & \quad \underbrace{\hspace{10em}}_{f\left(\frac{i}{n}\right)} \end{aligned}$$

Picture 3:

$$\begin{aligned} U(n) &= \sum_{i=1}^n f\left(\frac{i}{n}\right) \cdot \frac{1}{n} * \\ &= \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} \\ &= \sum_{i=1}^n \frac{i^2}{n^2} \cdot \frac{1}{n} \rightarrow \frac{n(n+1)(2n+1)}{6} \\ &= \sum_{i=1}^n \frac{i^2}{n^3} \end{aligned}$$

Picture 4:

$$\begin{aligned} U(n) &= \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \Big| \int_0^1 x^2 dx = \left(\frac{1}{3}\right) \\ &= \frac{2n^2 + 3n + 1}{6n^2} = U(n) \\ \lim_{n \rightarrow \infty} \left(\frac{2n^2 + 3n + 1}{6n^2}\right) &= \frac{2}{6} = \left(\frac{1}{3}\right) \end{aligned}$$

Notice we can write $\int_0^1 x^2 dx = \frac{1}{3}$ units².