MATH 166
Lesson 4.2a
The Definite Integral

In the previous section, we were setting up problems that looked like

$$\sum_{i=1}^{n}(\text{height}\cdot\text{width}) = \sum_{i=1}^{n} f\left(\tfrac{i}{n}\right)\cdot\tfrac{1}{n}.$$

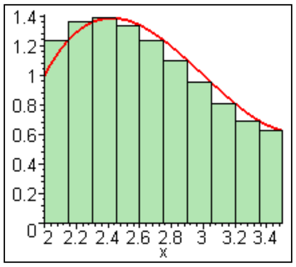
In theory, if we let $n \to \infty$, we get the exact area under the curve. We will use the notation Δx to stand for the width of the rectangle and x_i for the location on the *x*-axis to compute the corresponding heights. In other words, this sum is often expressed in the notation $\sum_{i=1}^{n}(\text{height}\cdot\text{width}) = \sum_{i=1}^{n} f(x_i)\Delta x$. Since the final step is to let *n* go to infinity, something should be said about the expression $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$. We know from the previous section that this rather formidable looking expression gives the **<u>area under the curve</u>** $y = f(x)$ on some prescribed interval $[a,b]$. Here is the formal definition:

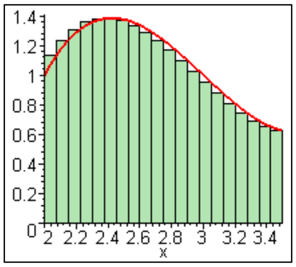
<u>Important Result</u>: Let $f(x)$ be continuous and nonnegative on $[a,b]$. Then the area bounded by $f(x)$, the *x*-axis, $x = a$, and $x = b$ is

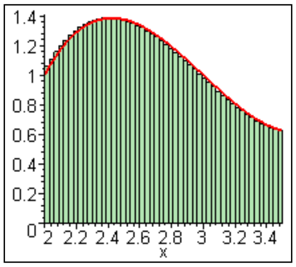
$$\boxed{\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x}$$

where x_i is **<u>ANY</u>** point in the i^{th} subinterval (it need not be a right or left endpoint) and $\Delta x = \dfrac{b-a}{n}$ (the width).

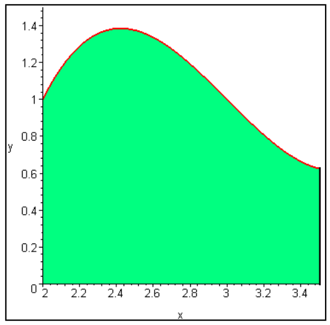
For example, pictures such as these will <u>not</u> give us the area because of the **<u>finite</u>** number of rectangles *n*:







However, if we let $n \to \infty$, we get the **<u>exact</u>** area:



We will soon abandon this summation notation and find a more efficient way to express these areas . . .

<u>Definition</u>: Let $f(x)$ be defined on $[a,b]$. If $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ exists, then we say that $f(x)$ is **integrable** on $[a,b]$ and we write

$$\boxed{\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx}.$$

We call $\int_a^b f(x)\,dx$ the **definite integral** of $f(x)$ on $[a,b]$. **The lower limit of integration** is *a* and the **upper limit of integration** is *b*.

<u>Note</u>: For now, this definition is simply telling us that we'll write $\int_a^b f(x)\,dx$ to mean "the area under the curve $y = f(x)$ on the interval $[a,b]$." We should be careful in doing this should $y = f(x)$ ever take on negative values. Regardless, this notation is definitely preferred over the cumbersome expression $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$.

Finally, here are some summation formulas; we will discuss them in detail in class.

<u>Theorem (Summation Formulas)</u>

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$