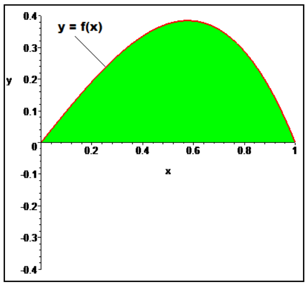
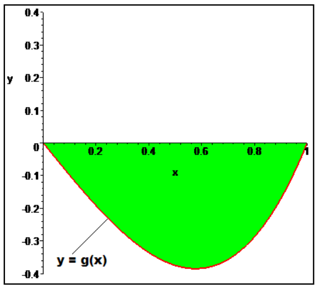
MATH 166
Lesson 4.2b
The Definite Integral

Another topic of discussion is the idea of net area (signed area). Notice that we have been very careful in always having a function above the *x*-axis. A natural question to ask is, "How is all of this interpreted if the function dips below the *x*-axis?"

> In general, the expression $\int_a^b f(x)\,dx$ gives a **net area** of the region bounded by the curve $y = f(x)$ and the *x*-axis on the interval $[a,b]$. The net area is the sum of the areas of the parts that lie above the *x*-axis minus the sum of the areas of the parts that lie below the *x*-axis on $[a,b]$.

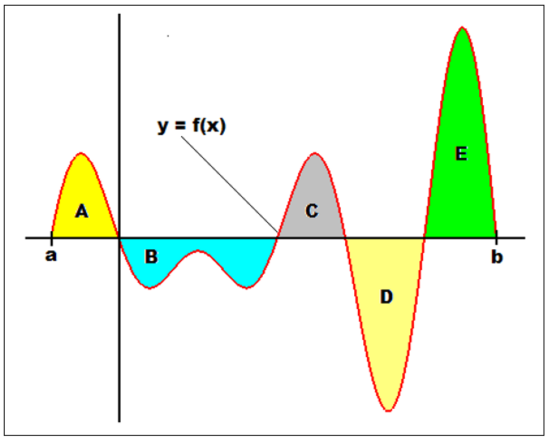
Example: Consider the shaded regions below. Note that $f(x) = -g(x)$.





Suppose you are given $\int_0^1 f(x)\,dx = 1/4$. Since $y = g(x)$ is the graph of $y = f(x)$ reflected over the *x*-axis, we conclude that $\int_0^1 g(x)\,dx = -1/4$.

Example: Consider the graph of $y = f(x)$ below.



Suppose the following facts are known:

$\text{Area}(A) = 6$
$\text{Area}(B) = 7$
$\text{Area}(A) = \text{Area}(C)$
$\text{Area}(D) = 11$
$\text{Area}(E) = 10$

Your task: find the value of $\int_a^b f(x)\,dx$.

Solution: Areas *A*, *C*, and *E* are *above* the *x*-axis so they can stay as is. The areas of regions *B* and *D* are *below* so they carry a negative sign. Thus, going in order from *A* to *E*, we get

$$\int_a^b f(x)\,dx = 6-7+6-11+10=4\,.$$

Below is a result stating some of the properties of definite integrals. Think about each property and see if you can give an interpretation of each (we'll do this in class).

Result:

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
4. $\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. $\int_a^b k\cdot f(x)\,dx = k\int_a^b f(x)\,dx$

Finally, here is a theorem that connects to some of our earlier work.

Theorem: If a function is continuous on the interval $[a,b]$, then it is integrable on $[a,b]$. In other words, we can find the area under the curve for continuous functions (or even for functions having a finite number of jump discontinuities).

Note: The theorem can be rephrased in the concise manner

$$\text{CONTINUITY} \Rightarrow \text{INTEGRABILITY}\,.$$

Combining this with the earlier theorem, $\text{DIFFERENTIABILITY} \Rightarrow \text{CONTINUITY}$ from way back when, we get the statement

$$\boxed{\text{DIFFERENTIABILITY} \Rightarrow \text{CONTINUITY} \Rightarrow \text{INTEGRABILITY}}\,.$$

This simple statement illustrates how continuous functions are at the heart of the operations of differentiation and integration. Much more on this later.