MATH 166
Lesson 4.4
Indefinite Integrals & The Net Change Theorem

This section discusses two important spin-offs from FTC II: $\int_a^b f(x)\,dx = F(b) - F(a)$.

First, in order to use FTC II, you must be able to find an antiderivative (or have access to a good supply of them). Since this lies at the heart of using the theorem, we address this first. It is traditional to use the notation $\int f(x)\,dx$ to indicate an antiderivative of $f(x)$.

For example, we learned in the last lesson that $\int_0^2 x^2 dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$. In this lesson, we will write $\int x^2 dx = \frac{x^3}{3} + C$. This latter form—called the **indefinite integral**—generates a family of antiderivatives (not a number like 8/3). Here is a basic table of indefinite integrals:

1. $\int 1\,dx = x + C$	2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
3. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$	4. $\int c\,f(x)dx = c\int f(x)dx$
5. $\int \sin x\,dx = -\cos x + C$	6. $\int \cos x\,dx = \sin x + C$
7. $\int \sec^2 x\,dx = \tan x + C$	8. $\int \csc^2 x\,dx = -\cot x + C$
9. $\int \sec x \tan x\,dx = \sec x + C$	10. $\int \csc x \cot x\,dx = -\csc x + C$

If you are comfortable with your derivatives, then the above table should be nothing new. For example, we know we can write $\frac{d}{dx}(\tan x) = \sec^2 x$ but another way of expressing this is through formula #7 above: $\int \sec^2 x\,dx = \tan x + C$. In general terms, we'll interpret $\int f(x)dx = F(x) + C$ as meaning $F'(x) = f(x)$.

The following distinction is an important one:

<u>The Definite Integral</u>
$\int_a^b f(x)\,dx$ is a **<u>number</u>** (it represents <u>area</u> if $f(x)$ is above the *x*-axis).

<u>The Indefinite Integral</u>
$\int f(x)\,dx$ is a **<u>family of functions</u>** (the antiderivatives of $f(x)$).

The second half of this lesson focuses on interpretations of $\int_a^b f(x)\,dx = F(b) - F(a)$ that are different from the "area under the curve". First, since $F'(x) = f(x)$ we write FTC II as $\int_a^b F'(x)\,dx = F(b) - F(a)$. Then depending on what the "rate" $F'(x)$ actually means, we sometimes obtain very rich interpretations:

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(1) An object moves along a straight line with position $s(t)$. The object's velocity is $v(t) = s'(t)$. Then $\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$ is the net change in position, or the **displacement** of the particle from $t = t_1$ to $t = t_2$.

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(2) If the rate of growth of a population is $N'(t)$ then $\int_{t_1}^{t_2} N'(t)\,dt = N(t_2) - N(t_1)$ is the **net change in the population** during the time period $t = t_1$ to $t = t_2$.

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(3) If $C(x)$ is the cost of producing x units of some commodity, then $C'(x)$ is the marginal cost. Then $\int_{x_1}^{x_2} C'(x)\,dx = C(x_2) - C(x_1)$, the **increase in cost** when increasing production from x_1 to x_2 units.

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(4) If a faucet leaks at the rate of $r(t)$ ounces per hour, then $\int_{t_1}^{t_2} r(t)\,dt =$ the number of ounces that have leaked from the faucet in $t_2 - t_1$ hours.

Just the small collection above explains why FTC II is sometimes called the **Net Change Theorem**. Besides the well-known area interpretation, $\int_a^b F'(x)\,dx = F(b) - F(a)$ calculates the net change in $F(x)$ when x changes from $x = a$ to $x = b$.