MATH 166
Lesson 4.5a
Substitution

In this section, we continue toward the goal of being able to integrate more functions (i.e., finding antiderivatives). If we think in terms of derivatives, we can say something like

$$\frac{d}{dx}\left[\tan\left(7x^2\right)\right] = 14x\sec^2\left(7x^2\right) \ . \tag{1}$$

If we reverse this statement, we can say

$$\int 14x\sec^2\left(7x^2\right)dx = \tan\left(7x^2\right) + C \, . \tag{2}$$

Taken together, the statements (1) and (2) are basically saying the same thing—one uses differentiation and the other uses integration. However, notice if you remove statement (1) and just look at the *problem* in statement (2), this is no longer a routine task. That is, consider the problem $\int 14x\sec^2\left(7x^2\right)dx$ and think about how you would do this problem without having access to the information in (1).

As you may have anticipated, this section is about the Chain Rule *in reverse* since we are integrating instead of differentiating. Here is an example that illustrates the basic idea.

Example: Evaluate $\int x^3\cos\left(x^4+2\right)dx$.

Solution: Let $u = x^4 + 2$ so that $\frac{du}{dx} = 4x^3$. Rearranging this, we get $\frac{du}{4} = x^3dx$. The motivation for this is that the term "x^3dx" shows up in the integrand. Here we go:

$$\begin{aligned}
\int x^3\cos\left(x^4+2\right)dx &= \int \cos\underbrace{\left(x^4+2\right)}_{u}\underbrace{x^3dx}_{du/4} \\
&= \int \cos u \cdot \frac{du}{4} \\
&= \frac{1}{4}\int \cos u \; du \\
&= \frac{1}{4}\sin u + C \\
&= \boxed{\frac{1}{4}\sin\left(x^4+2\right)+C}.
\end{aligned}$$

Checking the answer is a simple matter of differentiating the answer and making sure that it matches the integrand. Notice that $\frac{d}{dx}\left(\frac{1}{4}\sin\left(x^4+2\right)+C\right)=x^3\cos\left(x^4+2\right)$. A couple of notes are in order here.

Note 1: Just like in the Chain Rule for differentiation, we usually let "u = something in the problem" to simplify the integral. It is often the case that we choose something buried deep within the function (e.g., in the denominator, under a square root, inside parentheses, etc.). Sometimes, the choice of u depends heavily on what else can be seen in the integrand. This is the case for the example above.

Note 2: The goal is to transform the integral from one that appears difficult to one that is quite easy. In doing this, you end up switching to a new variable (usually u). After integrating, we eventually transform the problem back to its original variable.

Note 3: You may sometimes have to try one, two, or three substitutions before you find one that works. By "works" we mean it makes the new problem feasible. Sometimes a combination of skill and luck will accomplish this!

Here is another problem: $\int x\sqrt{x+1}\,dx$. Here we might try $u=x+1$ because then we get $\frac{du}{dx}=1$ or just $du=dx$. Looking back at the integral, we still need to get "x" but $u=x+1$ is equivalent to $x=u-1$. Using all of this information, we get

$$\underbrace{\int x\sqrt{x+1}\,dx}_{\text{Problem in } x}=\underbrace{\int (u-1)\sqrt{u}\,du}_{\text{Problem in } u}.$$

Although the integrals look similar, the one in variable u is quite a bit easier to finish. That is,

$$\begin{aligned}\int (u-1)\sqrt{u}\,du &= \int (u-1)u^{1/2}\,du\\ &= \int \left(u^{3/2}-u^{1/2}\right)du\\ &= \frac{u^{5/2}}{5/2}-\frac{u^{3/2}}{3/2}+C\\ &= \frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}+C\\ &= \boxed{\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+C}\end{aligned}$$

Thus, $\int x\sqrt{x+1}\,dx=\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+C$.