MATH 166
Lesson 4.5b
Substitution with Definite Integrals & Symmetry

Using *u*-substitution with a definite integral is not that different from using *u*-substitution with an indefinite integral but there are some subtle differences that are worth discussing. Below you will see the problem $\int_1^2 x\sqrt{x^2+1}\,dx$ worked out in two different ways. **<u>It is important to understand both methods but you can eventually choose the method you like better since they are equivalent.</u>**

<u>Method 1</u>: $\int_1^2 x\sqrt{x^2+1}\,dx$

We start with $u = x^2+1$ so $du = 2x\,dx$ or $\frac{du}{2} = x\,dx$. Next, notice that the lower limit of integration is $x=1$ and the upper limit is $x=2$. If we are truly going to transform this integral to the *u* variable, then this would mean changing the existing limits of integration ($x = 1 \to 2$) into new limits of integration ($u = ? \to ?$). We can use $u = x^2+1$ to figure this out (see the table below):

x (original limits of integration)	substitution $u = x^2+1$	*u* (new limits of integration)
1	$u = 1^2+1$	2
2	$u = 2^2+1$	5

Thus, our *u* limits of integration are $u = 2 \to 5$. The newly transformed problem looks like this: $\int_1^2 x\sqrt{x^2+1}\,dx = \int_1^2 \sqrt{x^2+1}\,x\,dx = \int_2^5 \sqrt{u}\,\frac{du}{2}$. Finishing, we have

$$\frac{1}{2}\int_2^5 u^{1/2}du = \frac{1}{2}\left.\frac{u^{3/2}}{3/2}\right|_2^5 = \boxed{\frac{1}{3}\left(5^{3/2}-2^{3/2}\right)}.$$

<u>Method 2</u>: $\int_1^2 x\sqrt{x^2+1}\,dx$

This method begins the same way with $u = x^2+1$ and $du = 2x\,dx$. At this point, you may notice that if the original integral had a "$2x\,dx$" the problem would actually be fairly easy to do. Since we only have "$x\,dx$" in the problem (and not "$2x\,dx$"), this is motivation to multiply the integral by 2—<u>but you have to also divide the integral by 2 so the net effect leaves the problem unchanged</u>. This looks like this:

$$\underbrace{\int_1^2 x\sqrt{x^2+1}\,dx}_{\text{original problem}} = \underbrace{\frac{1}{2}\int_1^2 \sqrt{x^2+1}\,(2x)\,dx}_{\text{problem rewritten}}$$

Since we now have a composite function $\sqrt{x^2+1}$ with the derivative of the inside function close by ($2x\,dx$), this means we can apply the Chain Rule in reverse. This looks like $\frac{1}{2}\int_1^2 \sqrt{x^2+1}\,(2x)\,dx = \frac{1}{2}\left.\frac{\left(x^2+1\right)^{3/2}}{3/2}\right|_1^2 = \left.\frac{1}{3}\left(x^2+1\right)^{3/2}\right|_1^2 = \boxed{\frac{1}{3}\left(5^{3/2}-2^{3/2}\right)}$.

Summative Note: Although **Method 2** encapsulates a bit more than **Method 1** (e.g., **Method 1** shows more detail), **Method 2** is preferred once you eventually get comfortable with substitution. The key advantage is that you don't need to actually switch to variable *u* in **Method 2**; this is true even for the limits of integration.

We close with some nice properties/shortcuts related to integrating even and odd functions from algebra. Recall that even functions have graphs that are symmetric with respect to the *y*-axis; odd functions have symmetry with respect to a 180 degree turn.

Symmetry Theorem:

1. If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
2. If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$.

Both of these have nice geometric interpretations. For f even, consider something like $\int_{-2}^{2} f(x)\,dx$ (see **Figure 1**). For f odd, consider something like $\int_{-1.8}^{1.8} f(x)\,dx$ (see **Figure 2**). Can you see why the Symmetry Theorem is true?

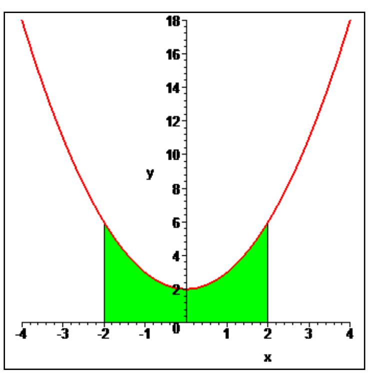

Figure 1. (f is even)

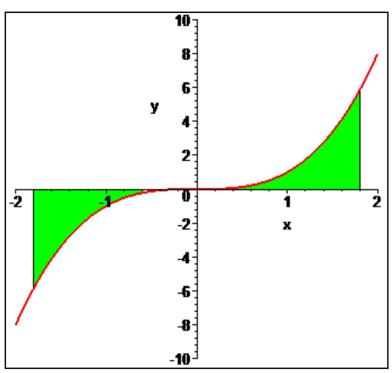

Figure 2. (f is odd)