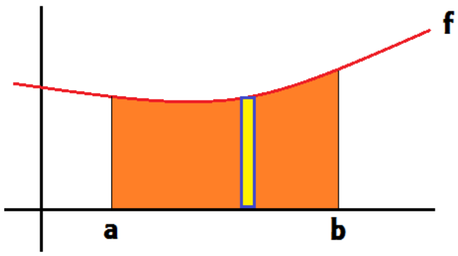
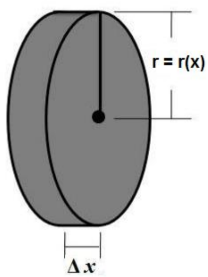
MATH 166
Lesson 5.2
Volume (Disc Method)

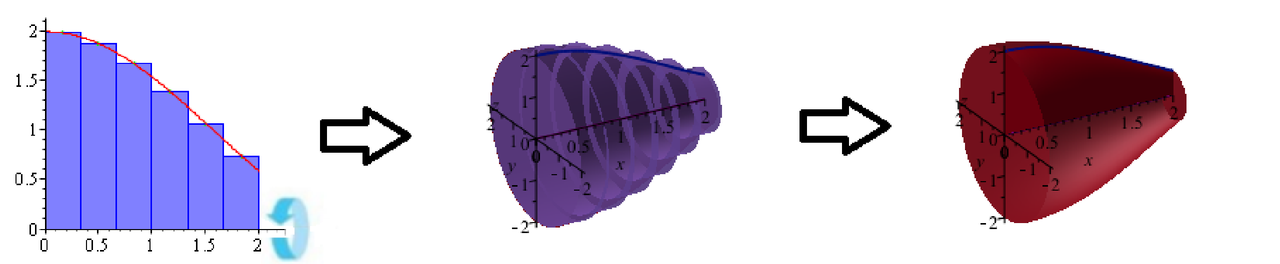
In the previous section, the topic was area; here, we will extend this idea to three dimensions so we will discuss the volume of solids. Start here:



Now suppose we revolve the shaded area about the *x*-axis (so we are using the *x*-axis like a hinge). This will generate a three-dimensional solid. Important here is that the representative rectangle in the above diagram becomes a disc:



Notice that the $r(x)$ on the disc is precisely the $f(x)$ from the graph. In other words, $r(x) = f(x)$. The volume of the above disc is equal to the area of the face, $\pi(r(x))^2$ times its thickness, Δx. Thus, the volume is $\pi(r(x))^2 \Delta x$. If we consider several of these disks on $[a,b]$ (for example, say $n = 6$ discs) then we would have to add their volumes and we would have the *approximate* volume of the solid: $\sum_{i=1}^{6} \pi(r(x_i))^2 \Delta x$. See this progression below:



If you used six rectangles (as just an example).....

....you would get six discs (not quite the desired volume)

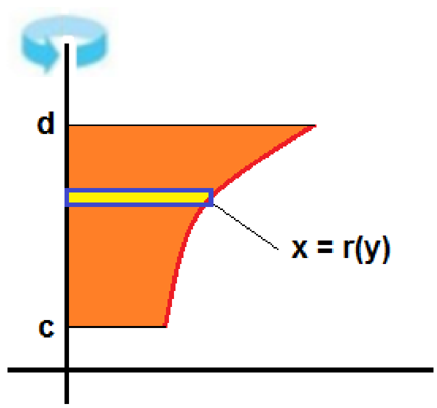
If you used an infinite number of rectangles, you'd get the exact volume

Considering an infinite number of rectangles, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}\pi\left(r\left(x_i\right)\right)^2\Delta x=\int_a^b\pi\left(r\left(x\right)\right)^2dx=\boxed{\pi\int_a^b\left(r\left(x\right)\right)^2dx}$$ ← Exact volume of 3D solid

This result is known as "the method of discs" or the "Disc Method." Notice its striking similarity to the geometry from which it was derived (the volume of a disc is $\pi\left(r\left(x\right)\right)^2\Delta x$ cubic units). The formulas are IDENTICAL but the Calculus formula takes into account the continuous accumulation (i.e., integration) on the interval $\left[a,b\right]$.

When revolving about a vertical axis, the situation is analogous:



In this case, we accumulate rectangles from *c* to *d*; these rectangles generate "stacked discs" of radius $r\left(y\right)$. The comparable formula is

$$\boxed{\text{Volume} = \pi\int_c^d\left(r\left(y\right)\right)^2dy}\ .$$

Two important notes:

(1) When you carefully observe both situations—rotating about a horizontal axis or rotating about a vertical axis—the key to using the disc method is that the representative rectangles are **perpendicular** to the axis of revolution. This is what fundamentally generates a **disc** and not some other shape.

(2) What happens when the rectangle does not "rest" on the axis of revolution? What does the 3D solid look like? What does the accompanying formula look like? We'll tackle this in class.