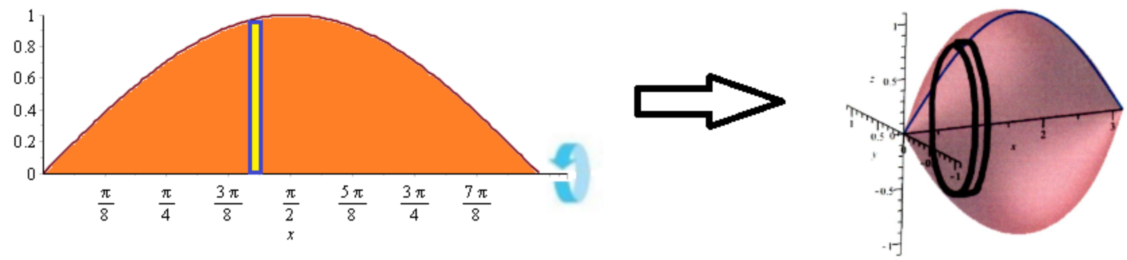
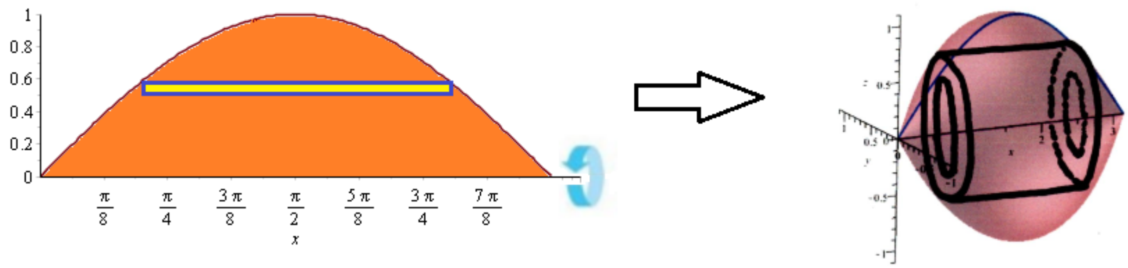
MATH 166
Lesson 5.3
Volume (Shell Method)

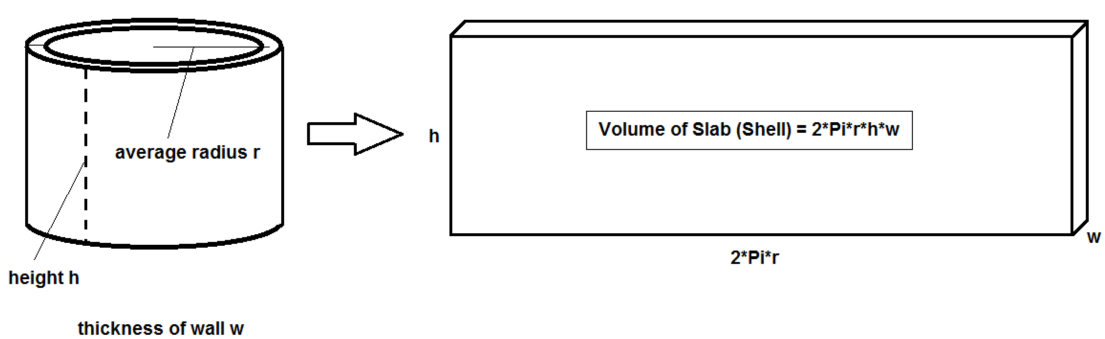
We continue the same topic as last lesson—the volume of a 3D solid of revolution—but we talk about a different way to calculate the volume. For example, consider the sine wave (and the area beneath it) on the interval $[0, \pi]$. Rotate this region about the *x*-axis to get a 3D solid:



Notice that the representative rectangle above becomes a disc (hence the name, the disc method). A different way to approach this is to change the orientation of the rectangle. Instead of using a rectangle perpendicular to the axis of revolution, we can try a **rectangle parallel to the axis of revolution**. This changes one fundamental thing: *What was once a disc is now a cylindrical shell.* See the diagram.



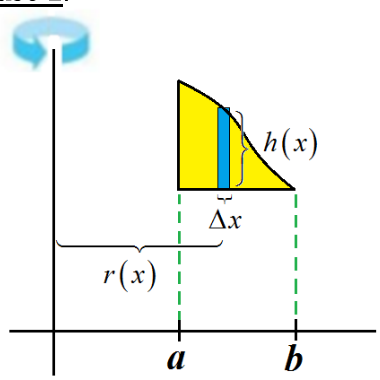
In the Shell Method, rather than "stacking discs" to generate the volume, the shells "nest" inside/outside one another to generate the volume. The one thing that doesn't change is the end result: We are still calculating the volume of the same 3D solid. It's just a different way (geometrically) to calculate it! If we study the shell shape carefully, we can see how to derive an integration formula. Slice the shell and lay it flat:



Just like the disc method $\pi\int_a^b (r(x))^2 dx$ is motivated by the volume of a disc $\pi(r(x))^2 \Delta x$ the shell method is motivated by the volume of a shell (this is easier to visualize if thought of as the volume of a rectangle with thickness *w*).

The two scenarios (with respect to *x* and with respect to *y*) are summarized below. Observe that the *w* becomes the differential (dx or dy) since the more rectangles you use, the slimmer the thickness of the shell (and the greater the accuracy toward the *exact* volume of the 3D solid).

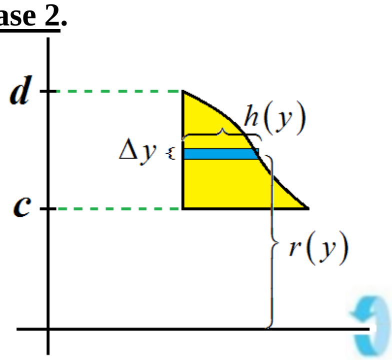
<u>Case 1</u>.



Notice the rectangle is parallel to the axis of revolution so it generates a cylindrical shell. The thickness of the shell is dx so integration is with respect to x. The radius of the shell is $r(x)$ and the height of the shell is $h(x)$. Using the geometric formula $\text{Volume} = 2\pi r(x)h(x)\cdot\text{thickness}$, the integration formula is

$\boxed{\text{Volume} = 2\pi\int_a^b r(x)h(x)dx}$. This gives the exact volume of the solid of revolution.

<u>Case 2</u>.



Using nearly identical language as in Case 1 (just switch variables from *x* to *y*), we get

$\boxed{\text{Volume} = 2\pi\int_c^d r(y)h(y)dy}$.