Theorem (Equivalent Conditions)

Let A be $n \times n$. The following statements are **equivalent**.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

(4) A is row equivalent to I_n.

(5) A can be expressed as the product of elementary matrices.

(6) $\det A \neq 0$.

(7) The columns of A are linearly independent.

TWO VECTORS

Is the set $\{\mathbf{u}, \mathbf{v}\}$ linearly independent?

$$\text{(a)}\quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

$$\text{(b)}\quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\text{(c)}\quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

EXAMPLE

Consider the vector space of all continuous functions on the real line, $C(-\infty,\infty)$. Show that the functions $\sin(2x)$ and $\sin x \cos x$ form a linearly dependent set in $C(-\infty,\infty)$.