DEFINITION

A linear transformation $T: V \to W$ that is one-to-one and onto W is called an **isomorphism**.

"isos" *equal*

"morph" *shape*

$$V \sim W$$

"V is isomorphic to W"

THE INVERTIBLE MATRIX THEOREM

Let A be $n \times n$. The following statements are **equivalent**.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b} \in \mathbb{R}^n$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

(4) A is row equivalent to I_n.

(5) A can be expressed as the product of elementary matrices.

(6) $\det A \neq 0$

(7) The columns of A are linearly independent.

(8) The columns of A span $\mathbb{R}^n$.

(9) The columns of A form a basis for $\mathbb{R}^n$.

(10) $\operatorname{col}(A) = \mathbb{R}^n$

(11) $\dim(\operatorname{col}(A)) = n$

(12) $\operatorname{rank}(A) = n$

(13) $\operatorname{nullity}(A) = 0$

(14) A does not have a zero eigenvalue.

(15) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$

(a) is one-to-one.

(b) maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(c) is an isomorphism.