

June 12, 2017

4.3: Fundamental Theorems of Calculus

PART 1:

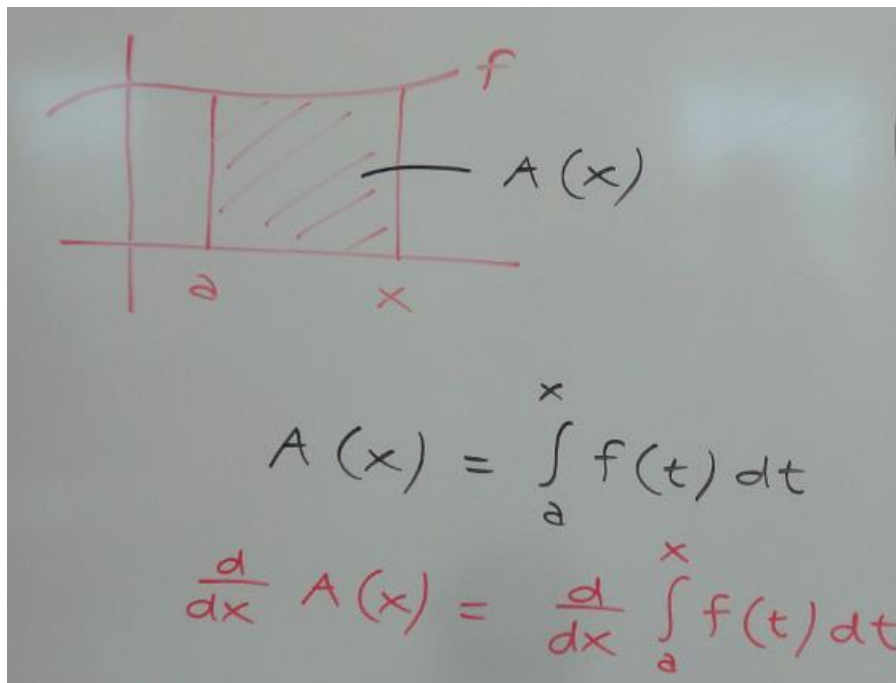
FUNDAMENTAL THEOREM; PART I

Let $f(x)$ be continuous on $[a, b]$.

If $A(x) = \int_a^x f(t) dt$,
then $A'(x) = f(x)$.

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Discussion of FTC I:



$$A'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x) = f(t) \Big|_{t=x}$$

Slope of TL: $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \frac{\Delta y}{\Delta x}$

A circular diagram shows the relationship between f , $\frac{d}{dx}$ (Slope of TL), and \int (Area).

Area:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \overbrace{f(x_i)}^{\text{height}} \overbrace{\Delta x}^{\text{width}}$$

A big point here is how integration (area) is a multiplication problem (height \times width) and differentiation (slope of the tangent line) is a division problem ($\Delta y / \Delta x$). This is one way of seeing integration and differentiation as inverse operations (similar to addition and subtraction).

PROBLEM

Find and graph the area function $A(x) = \int_2^x f(t) dt$.

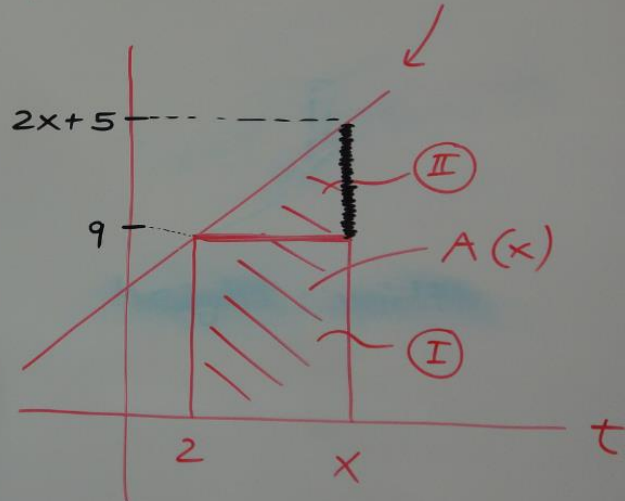
Then demonstrate FTC I by showing that $A'(x) = f(x)$.



$$A(x) = \int_2^x f(t) dt, \quad f(t) = 2t + 5$$

Show FTC I:

$$A'(x) = f(x)$$



$$A(x) = \int_2^x f(t) dt$$

$$= \textcircled{\text{I}} + \textcircled{\text{II}}$$

$$= 9(x-2) + \frac{1}{2}(x-2)(2x+5-9)$$

$$= 9x - 18 + (x-2)(x-2)$$

$$= 9x - 18 + x^2 - 4x + 4$$

$$= x^2 + 5x - 14$$

$$A(x) = x^2 + 5x - 14$$

$$A'(x) = 2x + 5$$

$$A'(x) = 2x + 5$$

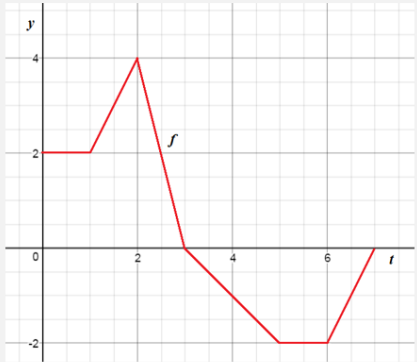
$$= f(x)$$

$$= f(t) \Big|_{t=x}$$

FTC I

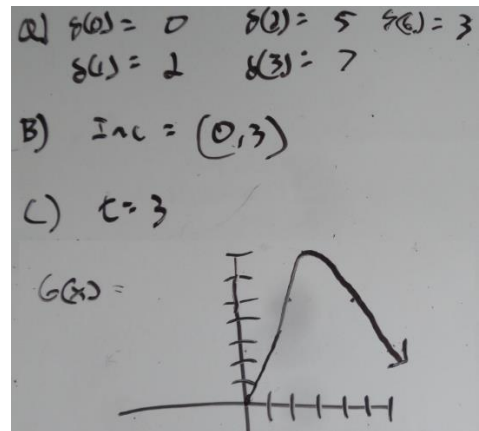
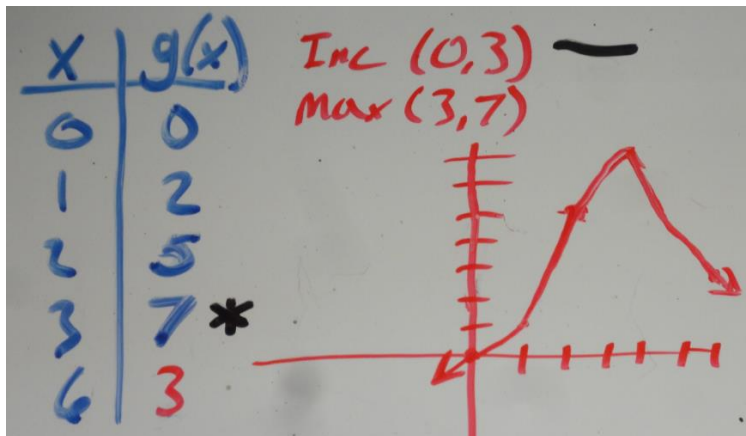
Boom! That's FTC I.

EXAMPLE



Let $g(x) = \int_0^x f(t) dt$.

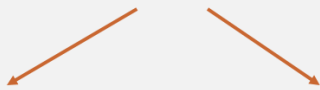
- (a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.
- (b) Where is g increasing?
- (c) Where does g have a maximum?
- (d) Sketch a rough graph of g .



PART 2:

FUNDAMENTAL THEOREM; PART II

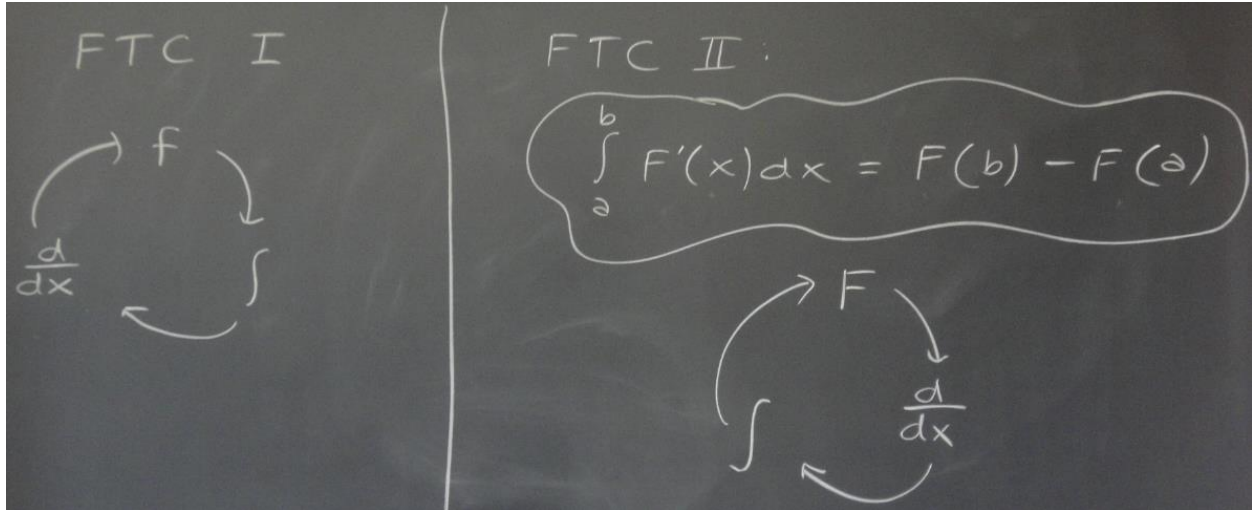
Let $f(x)$ be continuous on $[a, b]$.



$$\int_a^b f(x) dx = F(b) - F(a)$$

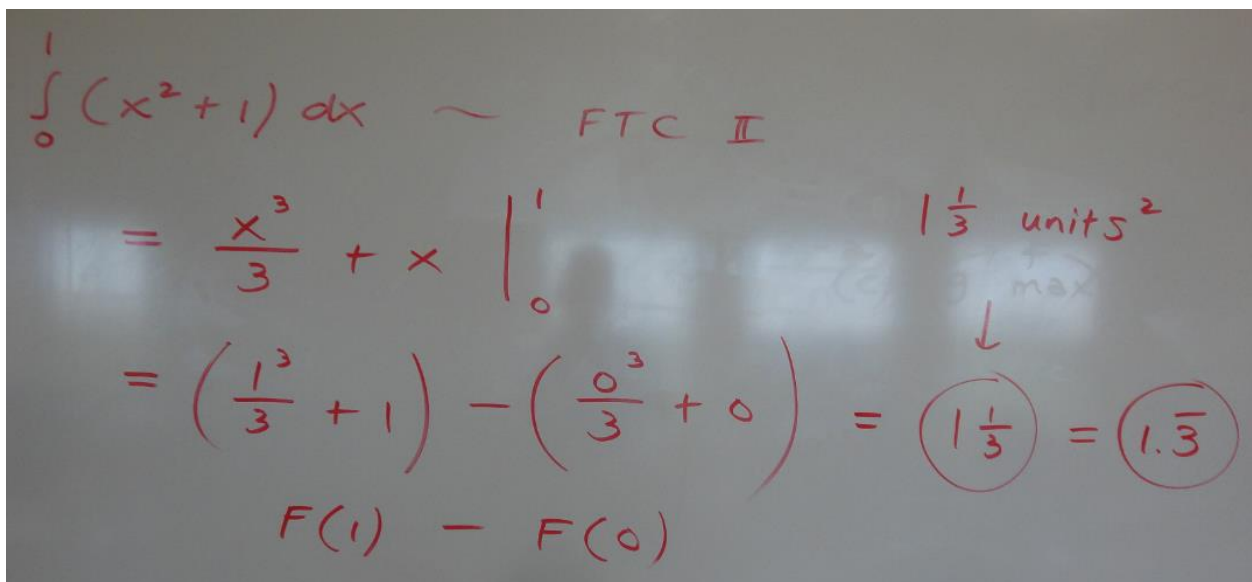
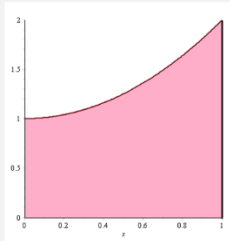
$$\int_a^b F'(x) dx = F(b) - F(a)$$

Compare/contrast the Fundamental Theorems:



PROBLEM

Determine the value of $\int_0^1 (x^2 + 1) dx$ by using the FTC II.



Recall:

$$1.21875 < \text{Area} < 1.46875$$

$\underbrace{\hspace{10em}}_{L(4)} \qquad \qquad \qquad \underbrace{\hspace{10em}}_{U(4)}$

The FTC II gives the **exact area** whereas the material from last week (upper sums/lowers sums) only gives an approximation (unless, of course, you let $n \rightarrow \infty$).

PROBLEMS

Evaluate the definite integrals. Use the FTC II if it applies.

(a) $\int_1^4 2\sqrt{x} \, dx$

(b) $\int_0^{2\pi} \cos t \, dt$

(c) $\int_0^{\pi/6} (1 - \sin \theta) \, d\theta$

(d) $\int_2^2 \frac{\sin x}{x} \, dx$

(e) $\int_1^4 (1-x)(x-4) \, dx$

(f) $\int_{-4}^1 |x| \, dx$

(A) $\int_1^4 2\sqrt{x} \, dx$

$F = \frac{4}{3}x^{3/2}$
 $F(4) - F(1) = 9\frac{1}{3}$

$2x^{1/2}$
 $\downarrow \int$
 $\frac{2}{3} \cdot 2x^{3/2}$

(B) $\int_0^{2\pi} \cos t \, dt$

$F = \sin t$

$\sin(2\pi) - \sin(0)$

$\int_0^{2\pi} \cos t \, dt = 0$

(C) $\int_0^{\pi/6} (1 - \sin \theta) \, d\theta$

$F = \theta + \cos \theta$

$(\frac{\pi}{6} + \cos \frac{\pi}{6}) - (0 + \cos 0)$

$(\frac{\pi}{6} + \frac{\sqrt{3}}{2}) - 1$ ← exact answer

$= .39$

(D) $\int_2^2 \frac{\sin x}{x} \, dx$

FTC II

$= 0 \rightarrow$ Upper bound + lower bound are the same, therefore this has no area.

(E) $\int_1^4 (1-x)(x-4) \, dx$

$x-4-x^2+4x$

$-x^2+5x-4$

$F(x) = -\frac{1}{3}x^3 + 2.5x^2 - 4x$

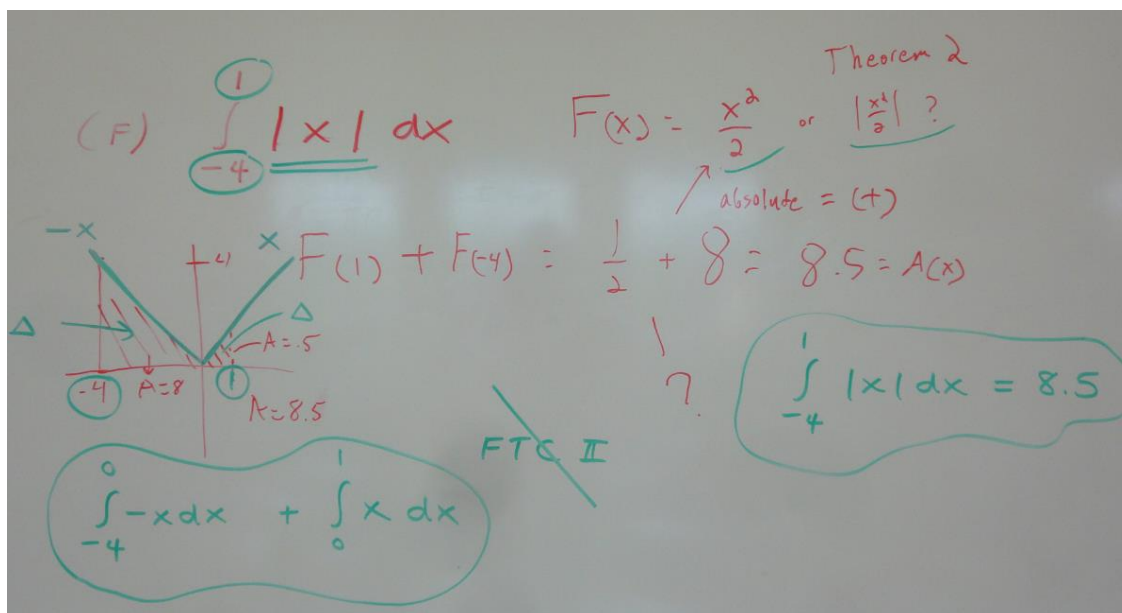
$F(4) - F(1)$

$2.6\bar{6} - -1.8\bar{3}$

$= 4.5$

$\rightarrow -\frac{x^3}{3} + \frac{5x^2}{2} - 4x \Big|_1^4$

$= \left(\frac{-64}{3} + \frac{5 \cdot 16}{2} - 16 \right) - \left(-\frac{1}{3} + \frac{5}{2} - 4 \right)$



Important notes: It is difficult/impossible to use FTC II on problem (d). What is the antiderivative of $\frac{\sin x}{x}$? Without an answer to this question, we don't get very far with the theorem.

It might be best to use a diagram and geometry to solve problem (f). The figures are triangles. Another option is to use the definition of absolute value $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$ and split the integral from $-4 \rightarrow 0$ and then $0 \rightarrow 1$ (see the board above).

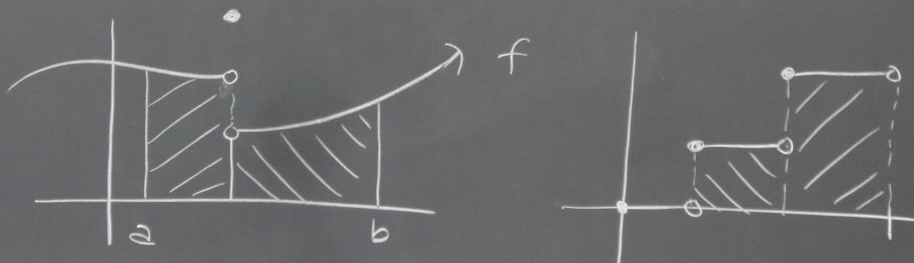
THINK ABOUT IT...

Under what circumstances would FTC II not apply? Give at least two examples.

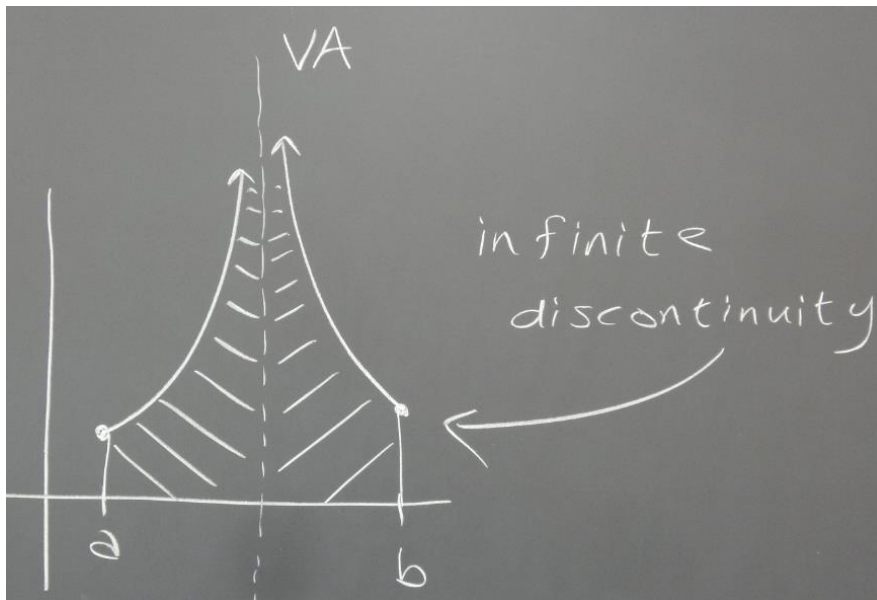
$$\int_a^b f(x) dx = F(b) - F(a)$$

When does FTC II
not apply?

- 1) Can't determine F
- 2) f is discontinuous



Note: In the discontinuous graphs above, “area” still seems well-defined. How about this?



Now we have a problem! How do you calculate the “area” in this case? We treat this type of problem in Calculus II.