June 12, 2017

4.3: Fundamental Theorems of Calculus

PART 1:

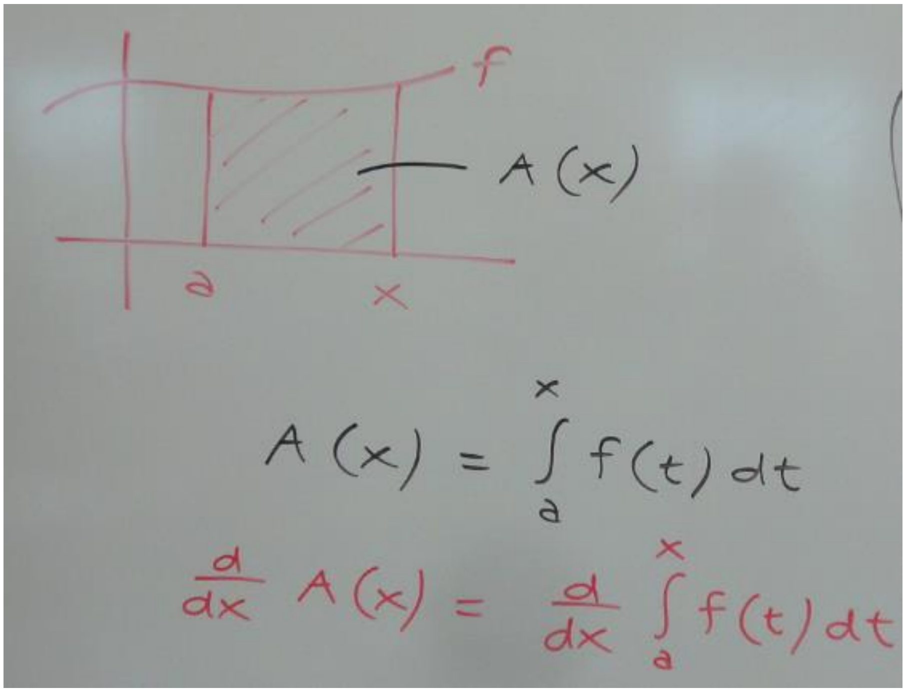
FUNDAMENTAL THEOREM; PART I

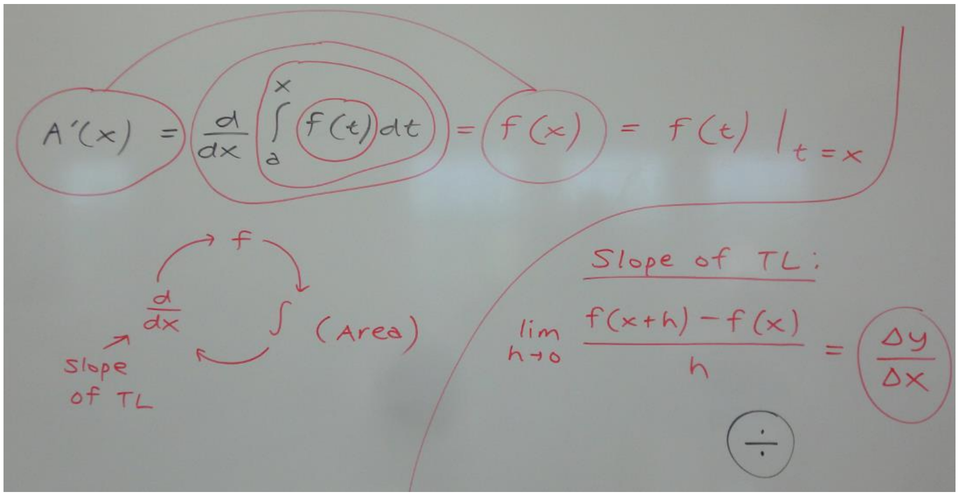
Let $f(x)$ be continuous on $[a,b]$.

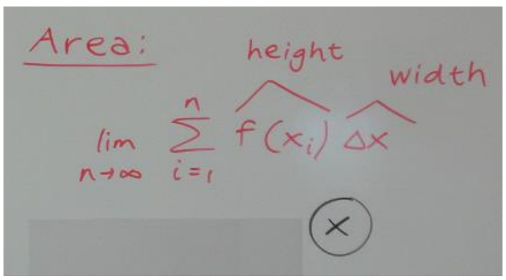
If $A(x)=\int_a^x f(t)\,dt$, then $A'(x)=f(x)$.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Discussion of FTC I:





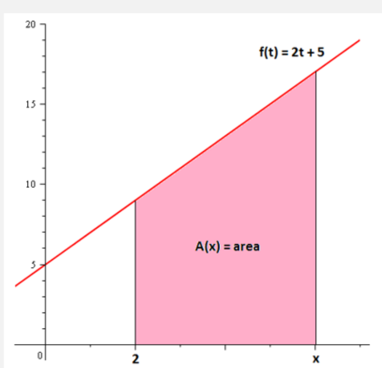


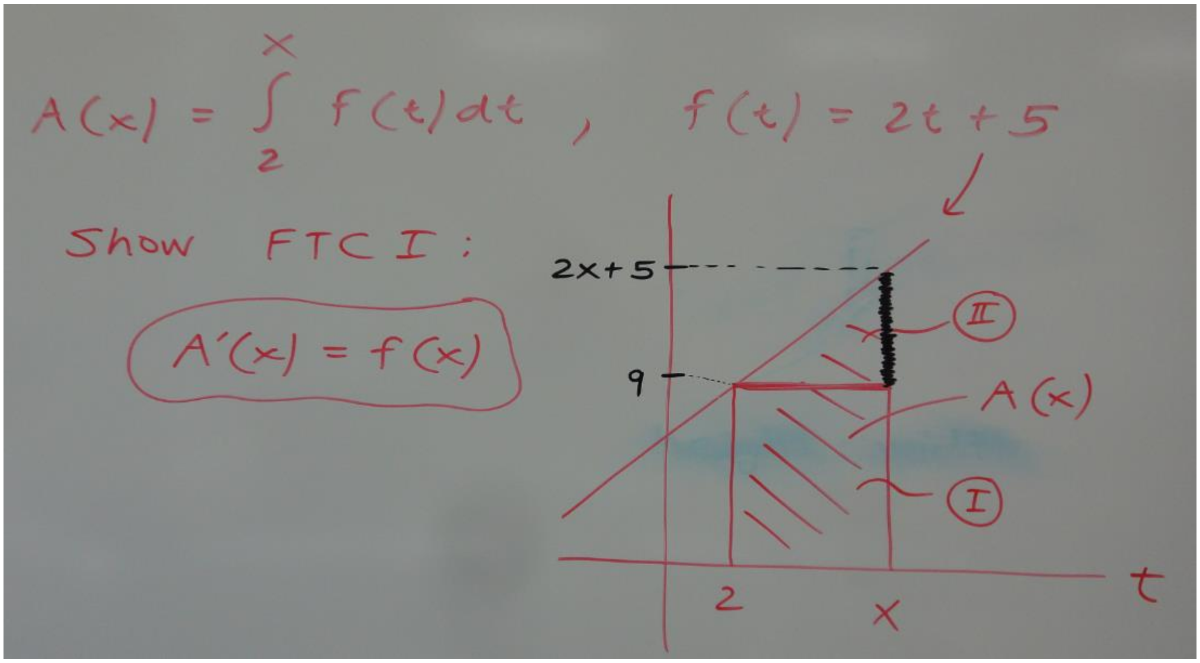
A big point here is how integration (area) is a multiplication problem (height $\times$ width) and differentiation (slope of the tangent line) is a division problem $(\Delta y / \Delta x)$. This is one way of seeing integration and differentiation as inverse operations (similar to addition and subtraction).

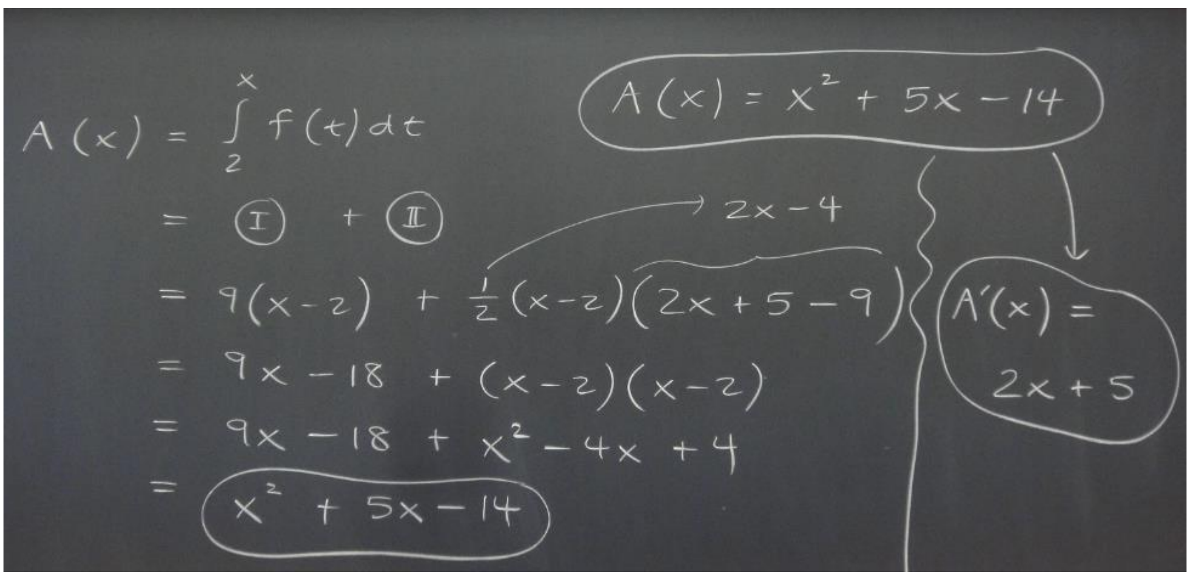
PROBLEM

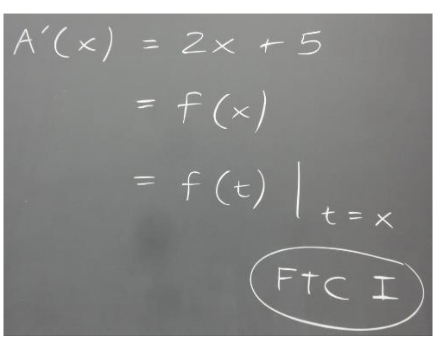
Find and graph the area function $A(x) = \int_2^x f(t)\,dt$.

Then demonstrate FTC I by showing that $A'(x) = f(x)$.



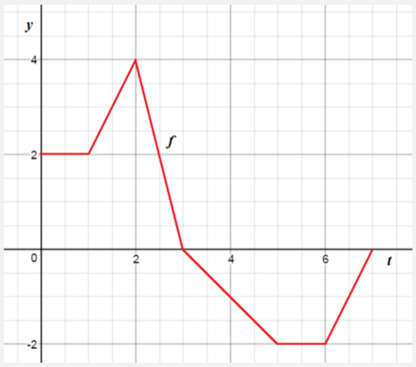






Boom! That’s FTC I.

EXAMPLE



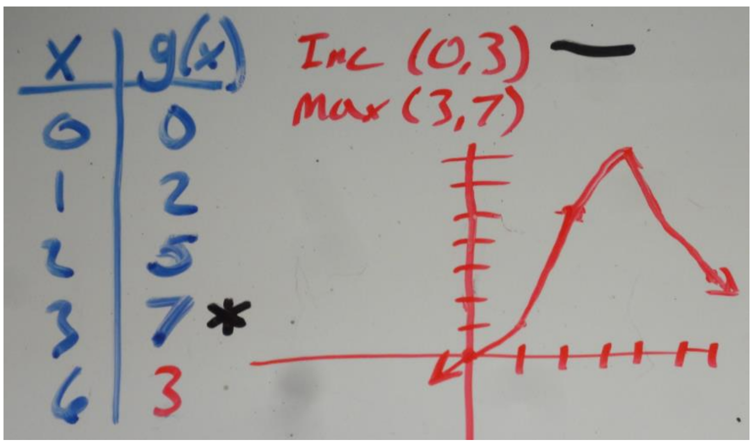
Let $g(x)=\int_0^x f(t)\,dt.$

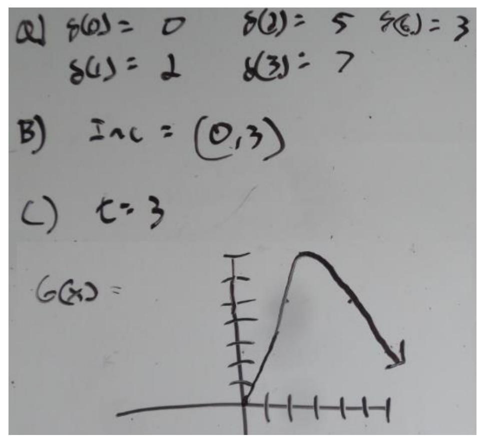
(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(6)$.

(b) Where is g increasing?

(c) Where does g have a maximum?

(d) Sketch a rough graph of g.





PART 2:

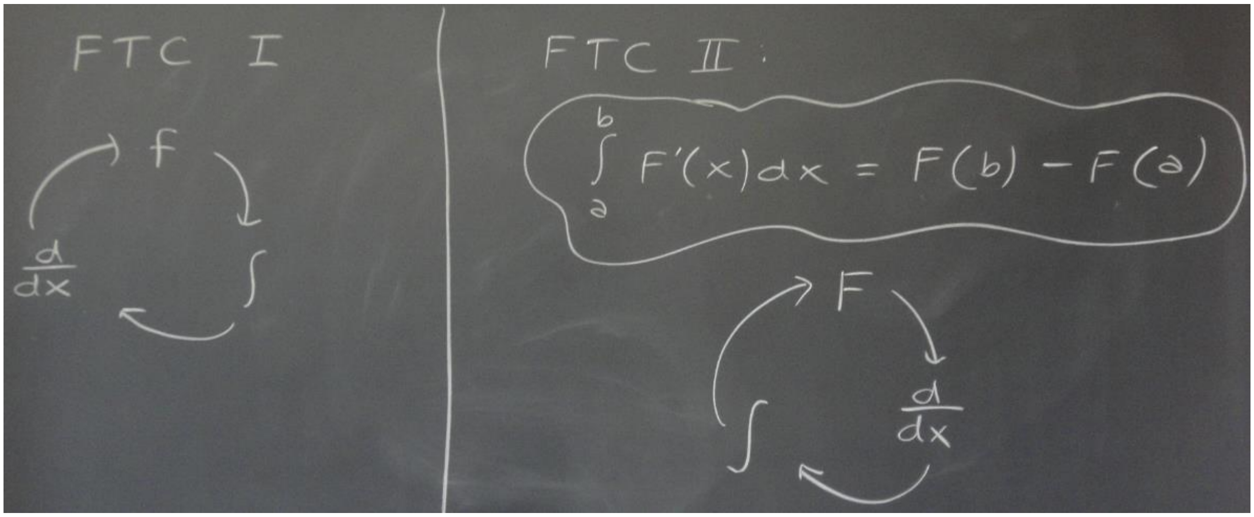
FUNDAMENTAL THEOREM; PART II

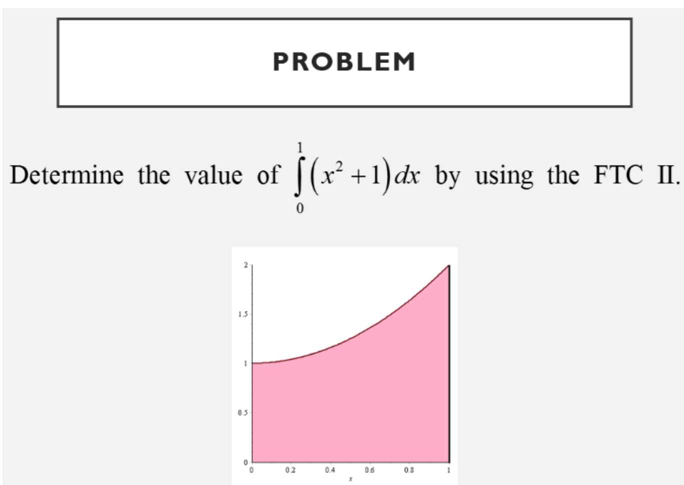
Let $f(x)$ be continuous on $[a,b]$.

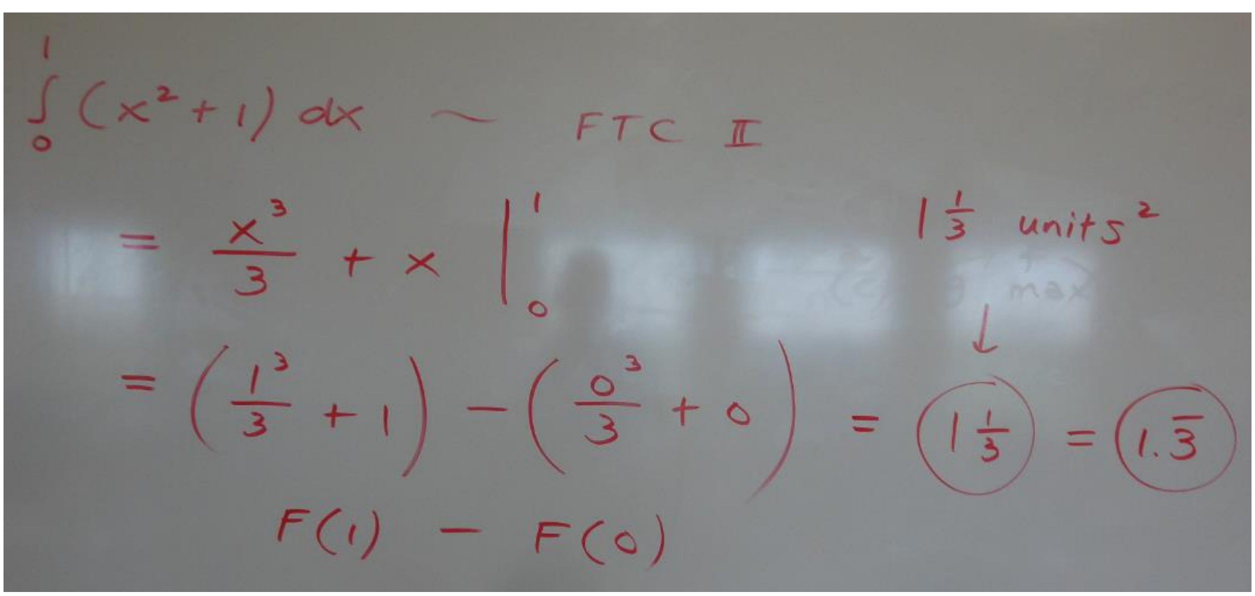
$$\int_a^b f(x)\,dx = F(b)-F(a)$$

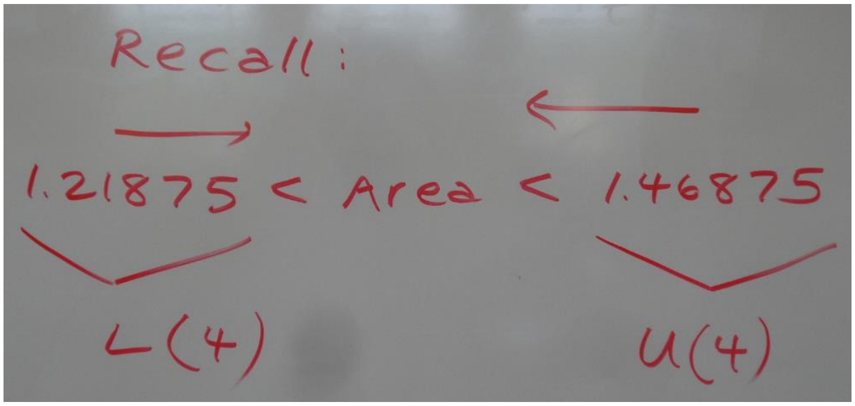
$$\int_a^b F'(x)\,dx = F(b)-F(a)$$

Compare/contrast the Fundamental Theorems:









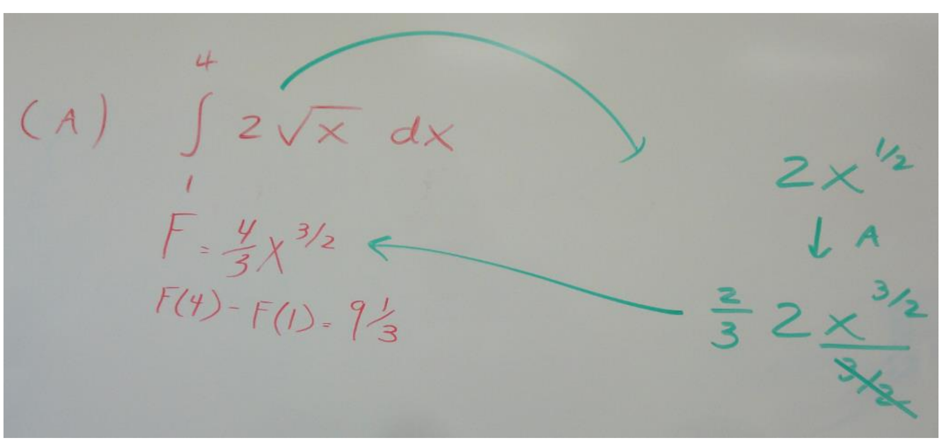
The FTC II gives the **exact area** whereas the material from last week (upper sums/lowers sums) only gives an approximation (unless, of course, you let $n \to \infty$).

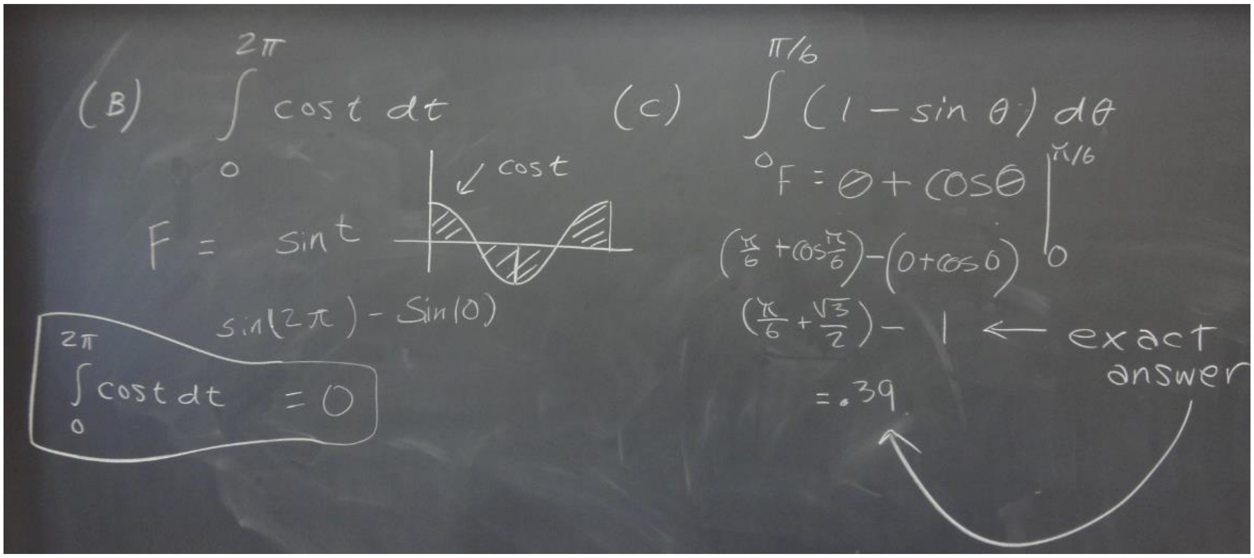
PROBLEMS

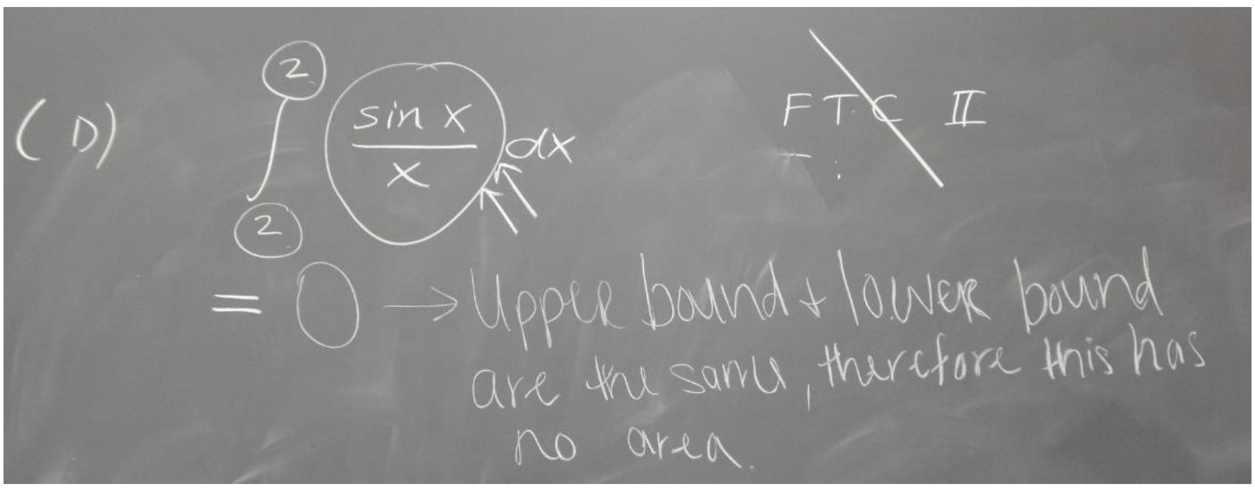
Evaluate the definite integrals. Use the FTC II if it applies.

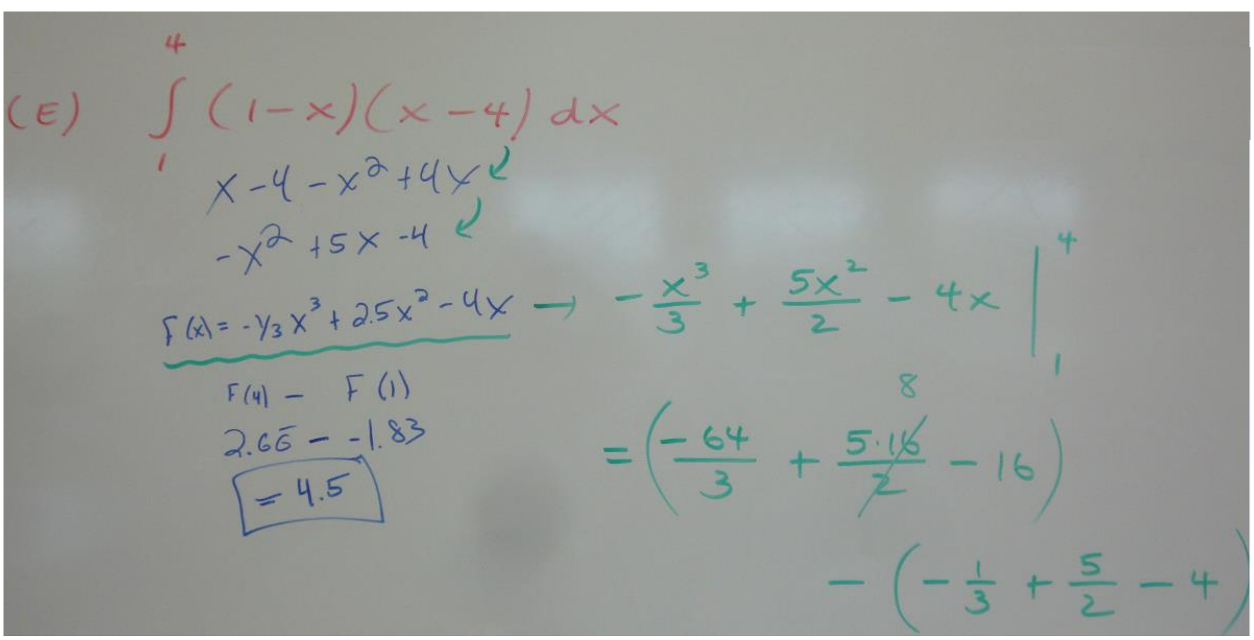
(a) $\displaystyle\int_1^4 2\sqrt{x}\,dx$ (b) $\displaystyle\int_0^{2\pi} \cos t\,dt$ (c) $\displaystyle\int_0^{\pi/6} (1-\sin\theta)\,d\theta$

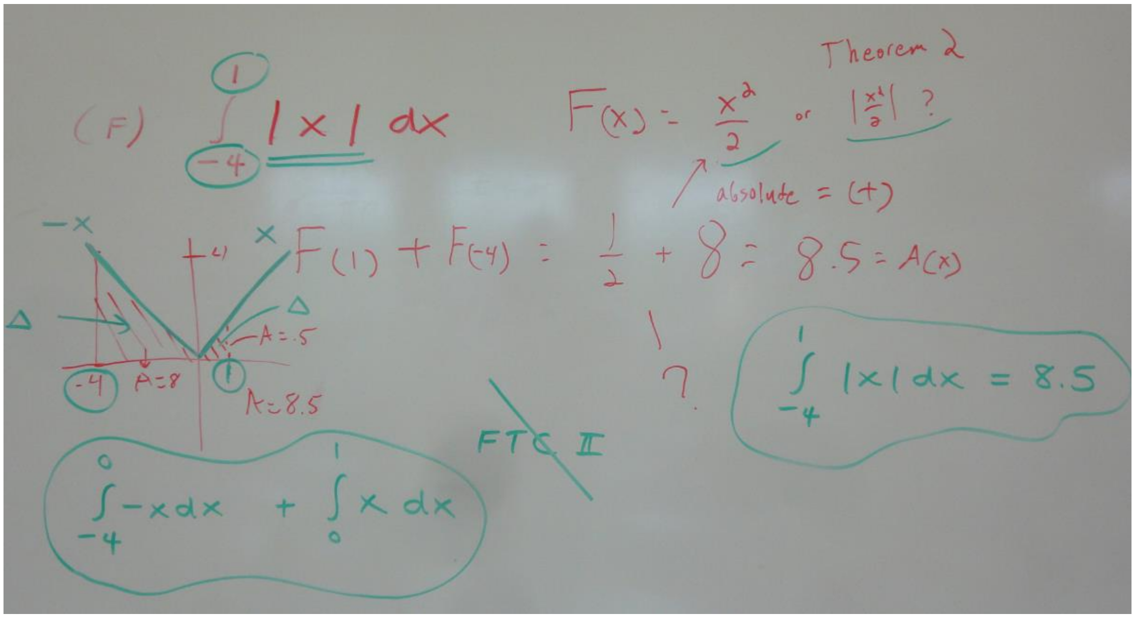
(d) $\displaystyle\int_2^2 \frac{\sin x}{x}\,dx$ (e) $\displaystyle\int_1^4 (1-x)(x-4)\,dx$ (f) $\displaystyle\int_{-4}^1 |x|\,dx$




(B) $\int_0^{2\pi} \cos t \, dt$
cos t
F = sin t
sin(2π) - Sin(0)
$\int_0^{2\pi} \cos t \, dt = 0$
(c) $\int_0^{\pi/6} (1 - \sin\theta) \, d\theta$
$F = \theta + \cos\theta \Big|_0^{\pi/6}$
$\left(\frac{\pi}{6} + \cos\frac{\pi}{6}\right) - (0 + \cos 0)$
$\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) - 1$ ← exact answer
= .39

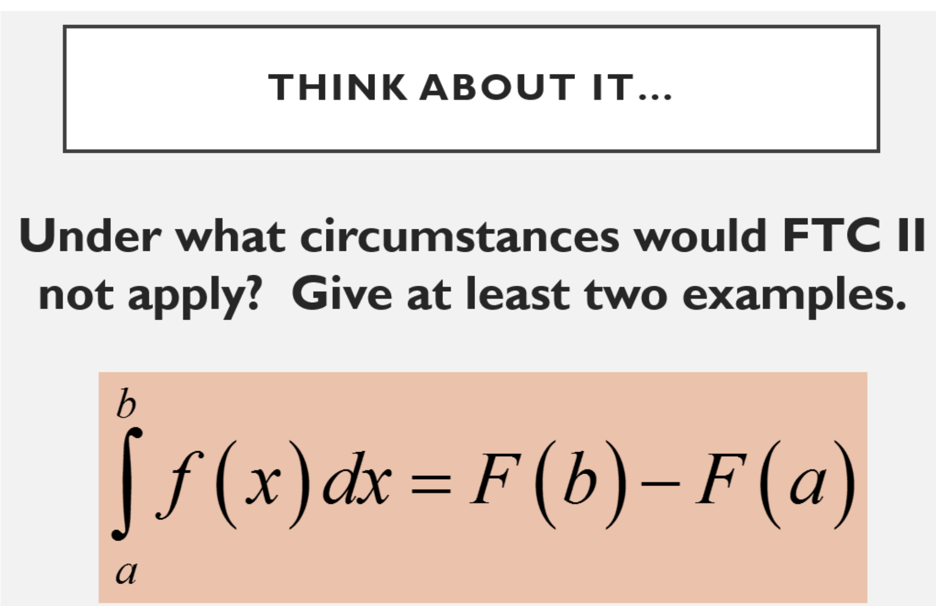

(D) $\int_2^2 \frac{\sin x}{x} \, dx$
FTC II
= 0 → Upper bound + lower bound are the same, therefore this has no area.

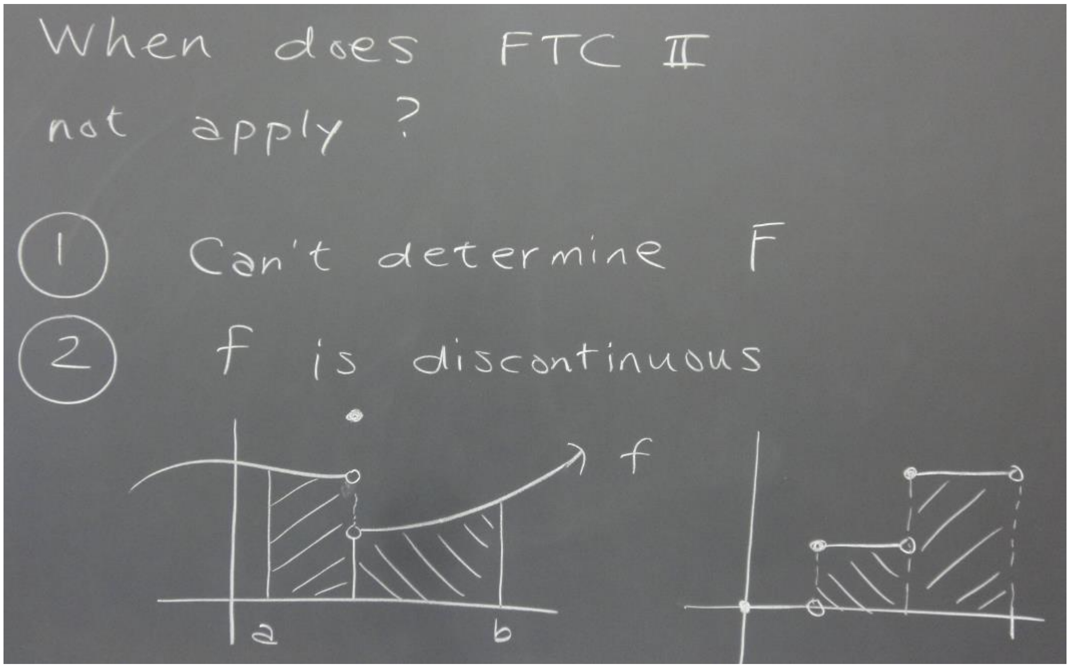

(E) $\int_1^4 (1-x)(x-4) \, dx$
$x - 4 - x^2 + 4x$
$-x^2 + 5x - 4$
$F(x) = -\frac{1}{3}x^3 + 2.5x^2 - 4x$ → $-\frac{x^3}{3} + \frac{5x^2}{2} - 4x \Big|_1^4$
F(4) - F(1)
2.66 - -1.83
= 4.5
$= \left(-\frac{64}{3} + \frac{5 \cdot 16}{2} - 16\right)$
$- \left(-\frac{1}{3} + \frac{5}{2} - 4\right)$



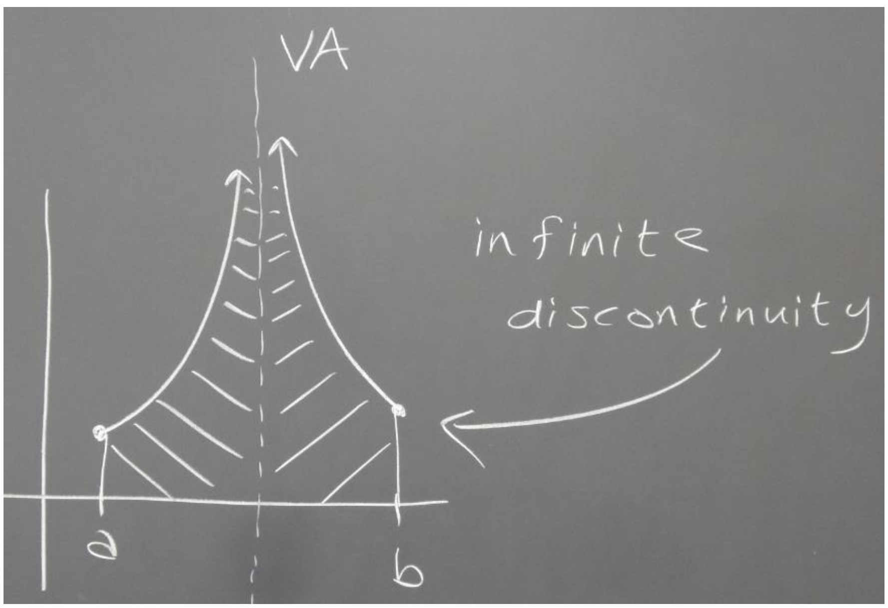
Important notes: It is difficult/impossible to use FTC II on problem (d). What is the antiderivative of $\frac{\sin x}{x}$? Without an answer to this question, we don't get very far with the theorem.

It might be best to use a diagram and geometry to solve problem (f). The figures are triangles. Another option is to use the definition of absolute value $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$ and split the integral from $-4 \to 0$ and then $0 \to 1$ (see the board above).





Note: In the discontinuous graphs above, "area" still seems well-defined. How about this?



Now we have a problem! How do you calculate the "area" in this case? We treat this type of problem in Calculus II.