

June 13, 2017

4.4: Indefinite Integrals & the Net Change Theorem

PROBLEM (LEVEL I)

Evaluate.

- (a) $\int x^{4/3} dx$ (b) $\int (t^2 + 4t + 5) dt$
(c) $\int \sec^2 \theta d\theta$

Handwritten solutions for Problem (Level I):

a) $\int x^{4/3} dx$
 $= \frac{3x^{7/3}}{7} + C$

b) $\int (t^2 + 4t + 5) dt$
 $= \frac{t^3}{3} + 2t^2 + 5t + C$

c) $\int \sec^2 \theta d\theta$
 $= \tan \theta + C$

PROBLEM (LEVEL II)

Evaluate.

- (a) $\int (3x + 5)^2 dx$ (b) $\int \frac{x^3 + 4x}{x} dx$
(c) $\int \sin(3x) dx$

$$A) \int (3x+5)^2 dx =$$

$$(3x+5)(3x+5) = \underline{9x^2 + 30x + 25}$$

$$F(x) = 3x^3 + 15x^2 + 25x + C$$

$$B) \int \frac{x^3 + 4x}{x} dy$$

$$= \int \left(\frac{x^3}{x} + \frac{4x}{x} \right) dx$$

$$= \int (x^2 + 4) dx$$

$$= \frac{1}{3}x^3 + 4x + C$$

$$C) \int \sin(3x) dx = \left(-\frac{1}{3} \cos 3x + C \right)$$

$$-\cos\left(\frac{3}{2}x^2\right) \leftarrow \text{incorrect}$$

$$\frac{d}{dx} \left(-\frac{1}{3} \cos(3x) \right) = \sin(3x)$$

$$\left. \begin{array}{l} \cos 3x \\ -\cos 3x \\ -\frac{1}{3} \cos 3x \end{array} \right\}$$

PROBLEM (LEVEL III)

Evaluate.

(a) $\int \tan^2 x \, dx$

(b) $\int \frac{1}{1 - \cos x} \, dx$

Problem A:

a) $\int \tan^2 x \, dx$
 $\int (\sec^2 x - 1) \, dx$
 $= \tan x - x + C$

The above notebook work is great as long as we recall the trig identity $\tan^2 x = \sec^2 x - 1$. What if you don't remember it? See below (using the definition of tangent along with the easier-to-remember identity $\sin^2 x + \cos^2 x = 1$ might be a better route). Either way leads to success.

$\int \tan^2 x \, dx = \int \left(\frac{\sin x}{\cos x} \right)^2 \, dx$
 $= \int \frac{\sin^2 x}{\cos^2 x} \, dx$
 $= \int \frac{1 - \cos^2 x}{\cos^2 x} \, dx$
 $= \int (\sec^2 x - 1) \, dx$
 $= \tan x - x + C$

Problem B:

(B) $\int \frac{1}{1-\cos x} dx$ $\xrightarrow{\text{I}}$ $\int (1-\cos x)^{-1} dx$?

$\swarrow \text{I}$ $\frac{1}{1} - \frac{1}{\cos x}$ (X)

$\searrow \text{III}$ $\int \frac{1}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x} dx$

$= \int \frac{1+\cos x}{1-\cos^2 x} dx$

$\rightarrow \underline{\underline{\sin^2 x}}$

(C) NTO

$\int \frac{1+\cos x}{\sin^2 x} dx$

$= \int \left(\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x} \right) dx$

$= \int \left(\csc^2 x + \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \right) dx$

$= \int \left(\csc^2 x + \csc x \cot x \right) dx =$

$-\cot x - \csc x + C$

FUNDAMENTALS

$$\int_a^b Q'(x) dx = Q(b) - Q(a)$$

[Net Change in Q]

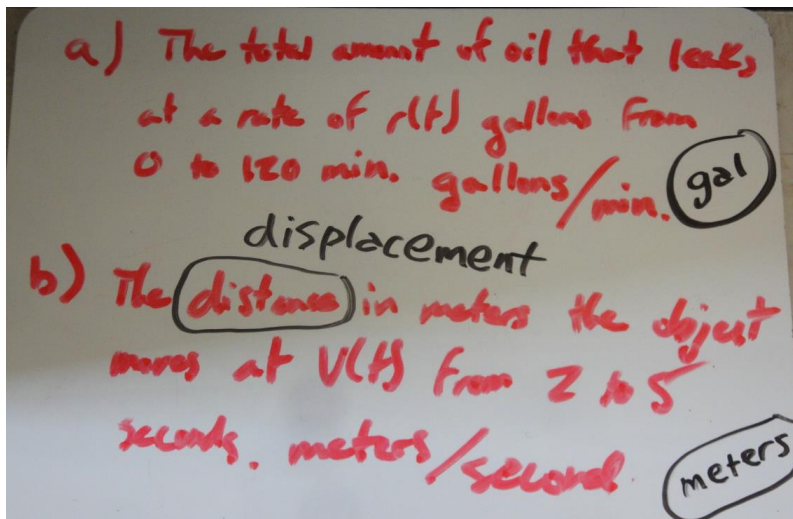
"THINKING" PROBLEMS

(a) Oil leaks from a tank at the rate of $r(t)$ gallons per minute at time t .

What is the meaning of $\int_0^{120} r(t) dt$? What are its units?

(b) If an object moves along a straight path with velocity $v(t)$ meters/sec,

what is the meaning of $\int_2^5 v(t) dt$? What are its units?



Supplemental work for part (b):

$$\begin{aligned} v(t) &: \text{velocity (m/s)} \\ \int_2^5 v(t) dt &= \int_2^5 s'(t) dt \\ &= s(5) - s(2) \\ &\text{displacement} \end{aligned}$$

4.5: Substitution

PROBLEMS

$$(a) \int \frac{x}{\cos^2(x^2)} dx$$

$$(b) \int \frac{9r^2}{\sqrt{1-r^3}} dr$$

Problem A:

Handwritten solution for Problem A on a chalkboard. The integral $\int \frac{x}{\cos^2(x^2)} dx$ is shown being transformed to $\int \frac{du/2}{\cos^2(u)}$, then $\frac{1}{2} \int \frac{du}{\cos^2 u}$. A purple sticky note with $\frac{1}{\cos^2 u} du$ is attached. The next steps are $\frac{1}{2} \int \sec^2 u du$, $\frac{1}{2} \tan u + C$, and finally $\frac{1}{2} \tan x^2 + C$. The substitution $u = x^2$ is marked with an asterisk, and the differential $\frac{du}{2} = x dx$ is circled.

$$\int \frac{x}{\cos^2(x^2)} dx = \int \frac{du/2}{\cos^2(u)} = \frac{1}{2} \int \frac{du}{\cos^2 u}$$

let $u = x^2$ *

$$\frac{du}{dx} = 2x$$
$$du = 2x dx$$
$$\frac{du}{2} = x dx$$
$$= \frac{1}{2} \int \sec^2 u du$$
$$= \frac{1}{2} \tan u + C$$
$$= \frac{1}{2} \tan x^2 + C$$

Problem B:

Handwritten solution for Problem B on a whiteboard. The integral $\int \frac{9r^2}{\sqrt{1-r^3}} dr$ is transformed to $9 \int \frac{du/(-3)}{\sqrt{u}}$, then $-3 \int \frac{du}{\sqrt{u}}$, $-3 \int u^{-1/2} du$, and $-3 \frac{u^{1/2}}{1/2} + C$. The final result is $-6\sqrt{u} + C$ and $-6\sqrt{1-r^3} + C$. The substitution $u = 1-r^3$ is marked with an asterisk, and the differential $\frac{du}{-3} = r^2 dr$ is circled.

$$(B) \int \frac{9r^2}{\sqrt{1-r^3}} dr = 9 \int \frac{du/(-3)}{\sqrt{u}}$$

$u = 1-r^3$ *

$$\frac{du}{dr} = -3r^2$$
$$du = -3r^2 dr$$
$$\frac{du}{-3} = r^2 dr$$
$$= -3 \int \frac{du}{\sqrt{u}}$$
$$= -3 \int u^{-1/2} du$$
$$= -3 \frac{u^{1/2}}{1/2} + C$$
$$= -6\sqrt{u} + C$$
$$= -6\sqrt{1-r^3} + C$$

Revisiting the problem $\int \sin 3x dx$, now with substitution:

The image shows a handwritten solution for the integral $\int \sin(3x) dx$. On the left side, the substitution $u = 3x$ is circled, followed by $\frac{du}{dx} = 3$, $du = 3 dx$, and $\frac{du}{3} = dx$ also circled. On the right side, the integral is solved step-by-step: $\int \sin(3x) dx = \int \sin(u) \frac{du}{3} = \frac{1}{3} \int \sin(u) du = \frac{1}{3} (-\cos u) + C = -\frac{1}{3} \cos(3x) + C$.

PROBLEMS

(a) $\int \sin^3 x \cos x dx$

(e) $\int \frac{\cos(\frac{1}{\theta})}{\theta^2} d\theta$

(b) $\int \sec(2z) \tan(2z) dz$

(f) $\int x \sqrt{9 - x^2} dx$

(c) $\int \sec^2 \theta \tan^4 \theta d\theta$

(g) $\int (3x + 4)^{10} dx$

(d) $\int v^5 \sqrt{1 + v^2} dv$

For each problem below, the step showing the transformation from the original variable (x, z, θ, v) to the u variable is circled/boxed. In each case, the problem in variable u is **easier** than the problem originally posed (and this is basically the point of u -substitution).

(A) $\int \overbrace{\sin^3 x}^{u^3} \overbrace{\cos x dx}^{du}$

$\int (\sin x)^3 \cos x dx$

$u = \sin x$
 $\frac{du}{dx} = \cos x$
 $du = \cos x dx$

$\int u^3 du$

$\frac{u^4}{4} + C$

$= \frac{\sin^4 x}{4} + C$

(B) $\int \sec(2z) \tan(2z) dz$

$u = 2z$
 $\frac{du}{dz} = 2$
 $du = 2 dz$
 $\frac{du}{2} = dz$

$\int \sec(u) \tan(u) \cdot \frac{du}{2}$

$\frac{1}{2} \int \sec(u) \tan(u) du$

$\frac{1}{2} \sec(u) + C$

$= \frac{1}{2} \sec(2z) + C$

(C) $\int \sec^2 \theta \tan^4 \theta d\theta$

$u = \tan \theta$
 $\frac{du}{d\theta} = \sec^2 \theta$
 $du = \sec^2 \theta d\theta$
 $\frac{du}{\sec^2 \theta} = d\theta$

$\int \sec^2 \theta (\tan \theta)^4 d\theta$

$\int \sec^2 \theta (u)^4 \frac{du}{\sec^2 \theta}$

$\frac{(u)^4}{5} \rightarrow \frac{(u)^5}{5} \rightarrow \frac{(\tan \theta)^5}{5} + C$

$\frac{1 + \tan^5 \theta}{5} + C$

$\int u^4 du$

(d) coming later...

$$(E) \int \frac{\cos(\frac{1}{\theta})}{\theta^2} d\theta$$

$$u = \frac{1}{\theta} \text{ or } \theta^{-1}$$

$$\frac{du}{d\theta} = -1\theta^{-2}$$

$$du = -1\theta^{-2} d\theta$$

$$-du = \theta^{-2} d\theta \text{ or } \frac{1}{\theta^2} d\theta$$

$$\int \frac{\cos(u)}{\theta^2} d\theta$$

$$\int \cos(u) \cdot -du$$

$$= -\sin(u) + C$$

$$= -\sin(\frac{1}{\theta}) + C$$

*

$$(F) \int x \sqrt{9-x^2} dx$$

$$u = 9-x^2$$

$$\frac{du}{dx} = -2x$$

$$du = -2x dx$$

$$\frac{du}{-2} = x dx$$

$$\int \sqrt{u} \left(\frac{du}{-2} \right)$$

$$-\frac{1}{2} \int \sqrt{u} du$$

$$-\frac{1}{2} \int u^{1/2} du$$

$$-\frac{1}{2} \frac{u^{3/2}}{3/2}$$

$$= -\frac{1}{3} \sqrt{u}$$

$$= -\frac{1}{3} \sqrt{9-x^2} + C$$

$$(G) \int (3x+4)^{10} dx$$

$$u = 3x+4$$

taking derivative

$$\frac{du}{dx} = 3$$

$$du = 3 \cdot dx$$

$$\frac{du}{3} = dx$$

moved $\frac{1}{3}$
took
integral
of
function
(leaving u
alone)

$$\int (u)^{10} \frac{du}{3}$$

$$\frac{1}{3} \frac{(u)^{11}}{11} + C$$

$$\frac{1}{3} \frac{(3x+4)^{11}}{11} + C$$

$$\frac{(3x+4)^{11}}{33} + C$$