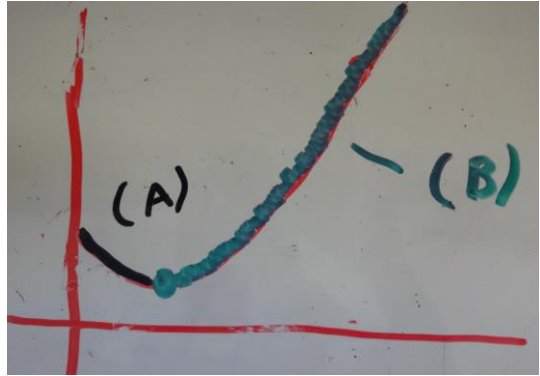


June 5, 2017

3.3: How Derivatives Affect the Shape of the Graph

WARM UP

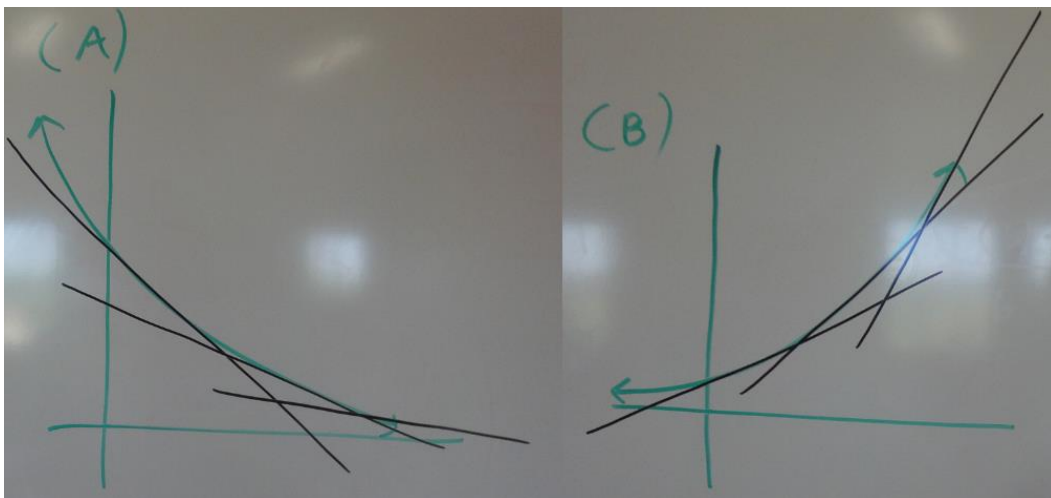
Consider a function $y = f(x)$ where f' is increasing. Sketch graphs of f such that (a) $f' < 0$ and (b) $f' > 0$.



The graph (above right) has both characteristics (see the labeling directly on the graph).

A great way to tackle a problem like this is to (a) translate/decode the meaning in the statements (see board below) and then (b) use that information to make a sketch.

Draw $y = f(x)$.
 f' is increasing \leftarrow slope is increasing
(A) $f' < 0$ (one example) \leftarrow slope is negative
(B) $f' > 0$ (a different example). \leftarrow slope is positive



PROBLEM

For each of the problems below, do the following:

- (a) Find the critical number(s).
- (b) State the intervals of increase/decrease.
- (c) Use the FDT to classify the extreme values.
- (d) Use your calculator to confirm your findings.

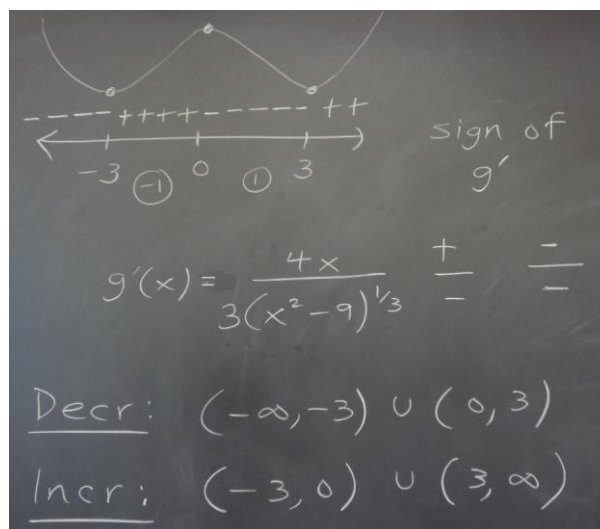
(a) $f(x) = x^2 - 4x$ (b) $g(x) = (x^2 - 9)^{2/3}$
 (c) $y = x\sqrt{6-x}$ (d) $h(x) = x + 2\sin x, 0 \leq x \leq 2\pi$

We did problem (b) together:

(b) $g(x) = (x^2 - 9)^{2/3}$
 $g'(x) = \frac{2}{3}(x^2 - 9)^{-1/3} \cdot 2x$
 $= \frac{4x}{3(x^2 - 9)^{1/3}}$ ↗ = 0
↘ DNE

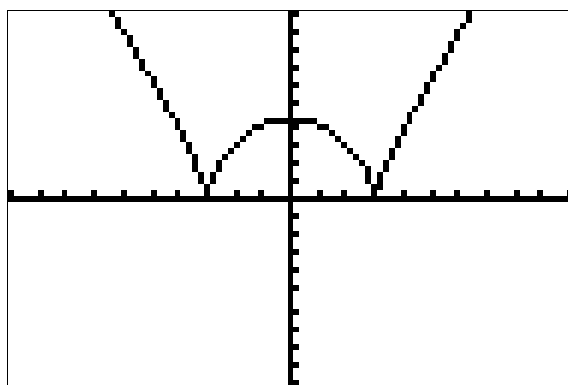
$x = 0$
 $x = \pm 3$

3 critical #'s



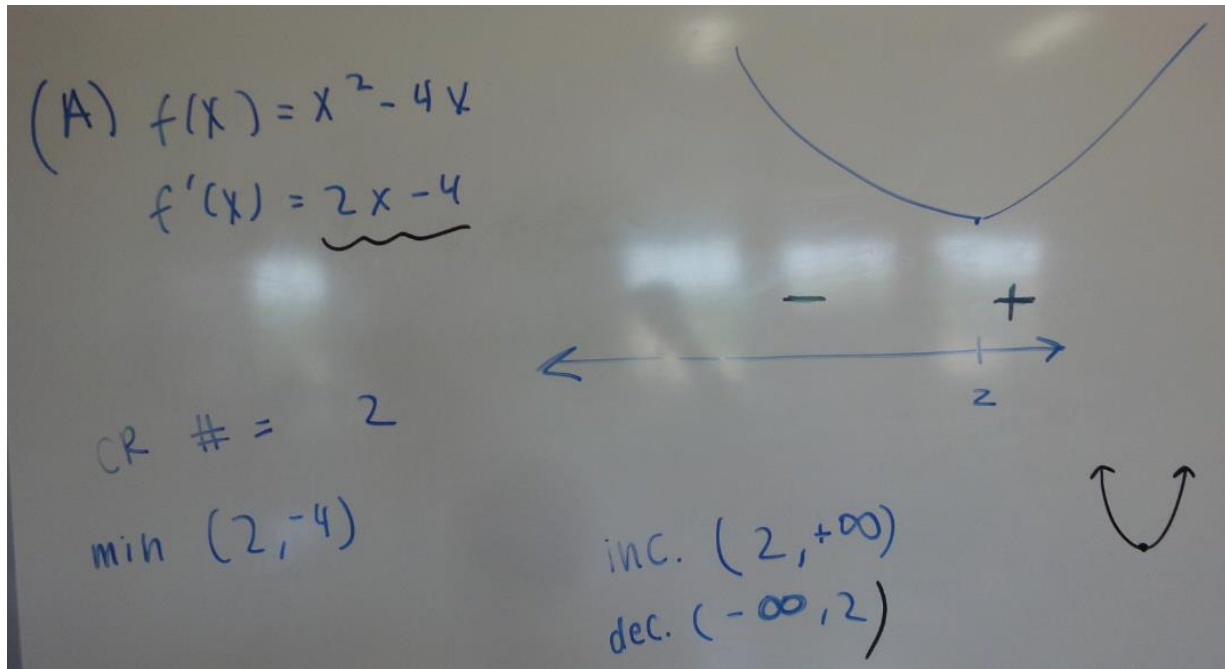
minimum: $(-3, g(-3))$
 $(3, g(3))$

maximum: $(0, g(0))$
 $\sqrt[3]{81}$

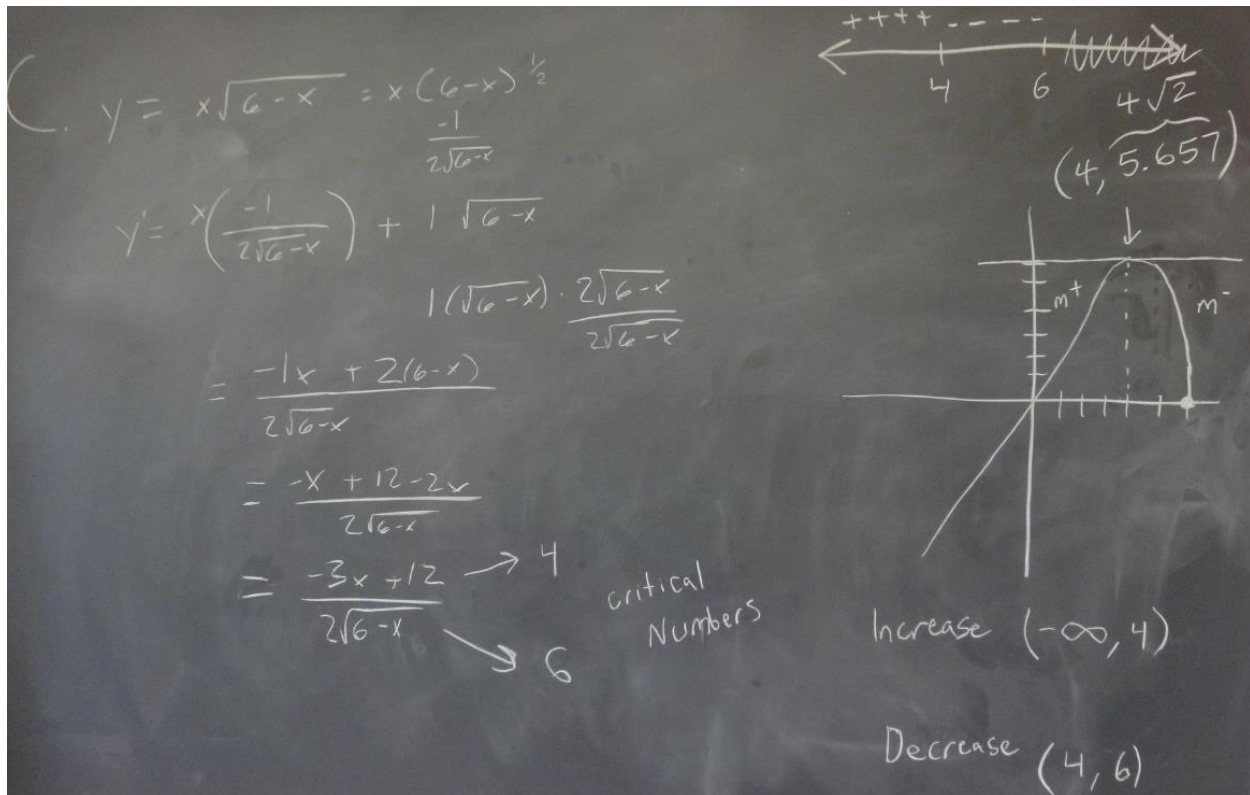


From this, we see minima at $(\pm 3, 0)$ and a maximum at $(0, \sqrt[3]{81})$. Notice the graph also shows the cusps at $(\pm 3, 0)$ —where $g(x)$ is not differentiable (not smooth).

Problem A:



Problem C:



Problem D:

(D) $h(x) = x + 2 \sin x$
 on $[0, 2\pi]$

$h'(x) = 1 + 2 \cos x \stackrel{\text{set}}{=} 0$

$2 \cos x = -1$
 $\cos x = -1/2$

critical #s

$x = \frac{\pi}{2} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

$x = \frac{3\pi}{2} - \frac{\pi}{6} = \frac{8\pi}{6} = \frac{4\pi}{3}$

INC DEC INC

++++ - - - + + + +

0 $\frac{2\pi}{3}$ $\frac{4\pi}{3}$ 2π Sign of $h'(x) = 1 + 2 \cos x$

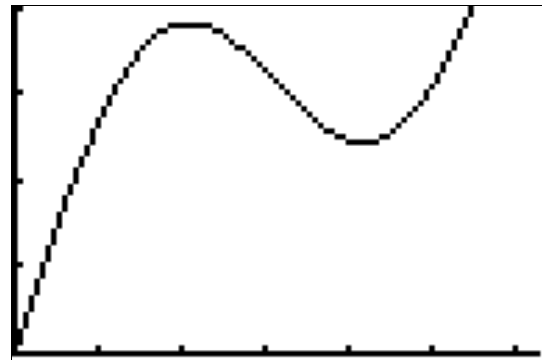
incr: $(0, \frac{2\pi}{3}) \cup (\frac{4\pi}{3}, 2\pi)$ ≈ 3.83

Decr: $(\frac{2\pi}{3}, \frac{4\pi}{3})$ \downarrow

max @ $x = \frac{2\pi}{3} \rightsquigarrow (\frac{2\pi}{3}, h(\frac{2\pi}{3}))$

min @ $x = \frac{4\pi}{3} \rightsquigarrow (\frac{4\pi}{3}, h(\frac{4\pi}{3}))$ \uparrow

≈ 2.45



Notice the above mathematical analysis identifies the hill and valley in the graph. The **absolute** min/max appear to be different from these values (absolute min at $(0,0)$ [lower left of the graph] and the absolute max at $(2\pi, 2\pi)$ [upper right of the graph]).

PROBLEM

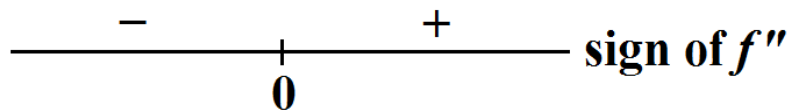
For each of the problems below, do the following:

- (a) Discuss the concavity of the graph.
- (b) Find any points of inflection.
- (c) Confirm the above on a calculator.

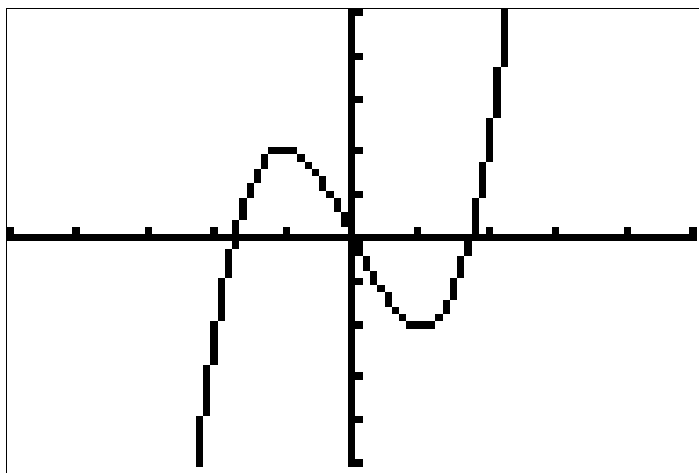
(a) $f(x) = x^3 - 3x$

(b) $y = x^4 - 4x^3$

In class we did Problem A, $f(x) = x^3 - 3x$. The second derivative is $f''(x) = 6x$; setting this to zero gives us $x = 0$. Therefore, $x = 0$ is a test value (a potential point of inflection). Using the usual number line analysis for f'' , we get this:



The graph of f is concave upward on $(0, \infty)$ and concave downward on $(-\infty, 0)$. We have an inflection point at $(0, f(0)) = (0, 0)$. See the graph:



Two versions of problem (b)...both look good:

$y = x^4 - 4x^3$
 $y' = 4x^3 - 12x^2$
 $y'' = 12x^2 - 24x$

CC \uparrow : $(-\infty, 0) \cup (4, \infty)$
 CC \downarrow : $(0, 2)$
 P.O.I.: $x=0, x=2$
 $(0, 0)$ & $(2, -16)$

$12x(x-2)$
 $12x = 0 \quad x = 0$
 $x-2 = 0 \quad x = 2$

$y = x^4 - 4x^3$
 $y' = 4x^3 - 12x^2$
 $y'' = 12x^2 - 24x$

$x=0$
 $x=2$

CC \downarrow $(0, 2)$
 CC \uparrow $(-\infty, 0) \cup (2, \infty)$
 P.O.I.: $(0, 0)$ $(2, -16)$

PROBLEM

For each of the problems below, do the following:

- Identify the critical number(s).
- Discuss the concavity of the graph.
- Use the **Second Derivative Test** to classify the extreme value(s). See part (a).
- Confirm the above on a calculator.

(a) $f(x) = x^3 - 3x^2 + 3$

(b) $y = x + 1/x$

$$y = x^3 - 3x^2 + 3$$

$$y' = 3x^2 - 6x = 0$$

↑
set

$$3x(x-2) = 0$$

$x = 0, 2$ ~ critical #s

$$y' = 3x^2 - 6x$$

$$y'' = 6x - 6 = 0$$

↑
set

$x = 1$

----- + + + +
←-----|-----→ sign of y''
CC↓ CC↑

CC↓ : $(-\infty, 1)$
CC↑ : $(1, \infty)$ } (B)

$$y'' = 6x - 6$$

$x = 0, 2$ ← critical #s

$$y''(0) = 6(0) - 6 = -6 < 0$$

$$y''(2) = 6(2) - 6 = 6 > 0$$

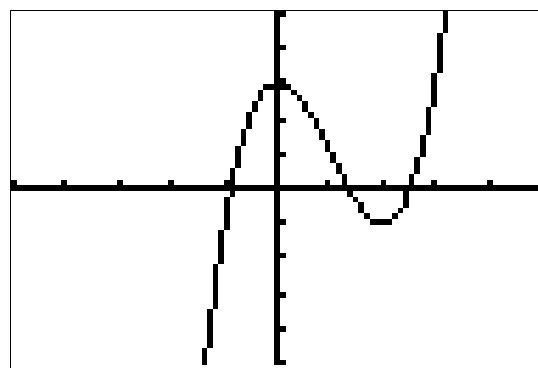
CC↓
CC↑

3

$(0, y(0))$ max

$(2, y(2))$ min

-1



So here we figure out that $(0, 3)$ is a maximum and $(2, -1)$ is a minimum but this is done **by examining the concavity of the graph.**

Problem B:

Nabb's work:

$y = x + \frac{1}{x} \quad x \neq 0$

$y' = 1 - \frac{1}{x^2}$

$y'' = -(-2/x^{-3})$
 $= \frac{2}{x^3}$

$1 - \frac{1}{x^2} = 0$
 $x = \pm 1$
critical x

just $x \neq 0$

sign of y''

CC ↓ $(-\infty, 0)$
CC ↑ $(0, \infty)$

$y'' = \frac{2}{x^3}$
 $y''(1) = 2 > 0$
 $y''(-1) = -2 < 0$

$(1, f(1))$ MIN
 $(-1, f(-1))$ MAX

$(1, 2)$ MIN
 $(-1, -2)$ MAX

