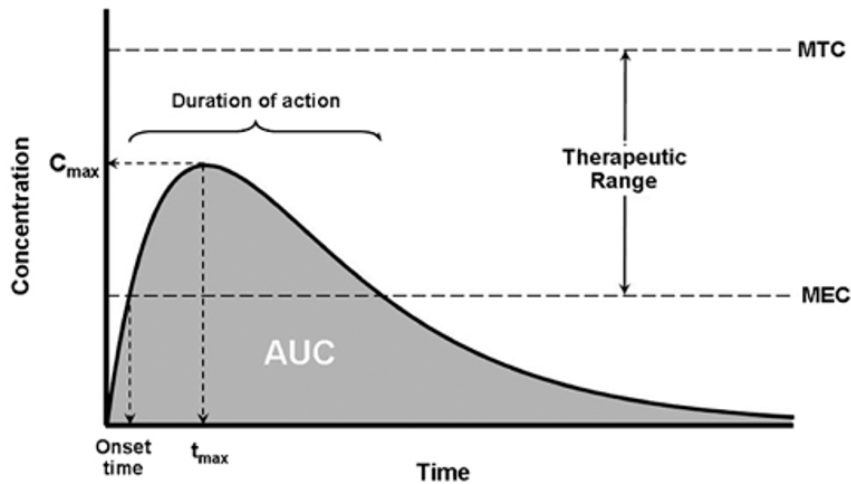


June 6, 2017

### 3.4: Limits At Infinity; Horizontal Asymptotes

## DRUG CONCENTRATION IN THE BLOODSTREAM AS A FUNCTION OF TIME



Source: medscape.com

## PROBLEM

Study the following three limits.

(a)  $\lim_{x \rightarrow \infty} \frac{2x^2 - 4}{3x^2 + x}, L = \frac{2}{3}$

(b)  $\lim_{x \rightarrow \infty} \frac{2x^2 - 4}{3x^3 + x}, L = 0$

(c)  $\lim_{x \rightarrow \infty} \frac{2x^3 - 4}{3x^2 + x}$ , limit does not exist.

Explain how these limits can be determined by inspection.

Answering these questions informally:

- a) exponents =
- b) leading coefficients bottom is bigger so you'll continue to divide by larger numbers
- c) top coefficient larger it will continue to grow

Showing all the details for part (a):

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 4}{3x^2 + x} \div x^2 \div x^2$$

(ANS:  $\frac{2}{3}$ )

$$= \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} - \frac{4}{x^2}}{\frac{3x^2}{x^2} + \frac{x}{x^2}} =$$

$$\lim_{x \rightarrow \infty} \frac{2 - \frac{4}{x^2}}{3 + \frac{1}{x}}$$

$x \rightarrow \infty$

$$= \lim_{x \rightarrow \infty} \frac{2 - \frac{4}{x^2}}{3 + \frac{1}{x}} \rightarrow \frac{2}{3}$$

**PROBLEM**

Evaluate  $\lim_{x \rightarrow \infty} \frac{\sin x}{x}$ .

Compare/contrast this with the problem  $\lim_{x \rightarrow \infty} \sin x$ .

x	y	x	y
1	0.8414	800	0.001
10	0.0541		
100	-0.005		
200	-0.0043		
400	-0.00212		

Handwritten notes on the table: "x=1" above the first row, "lim\_{x \to \infty} = 0" written in a circle on the right side.

A great way to start this problem is to look at a table (see board, upper right). From the table, it appears that  $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$ . Notice, on the other hand, that  $\lim_{x \rightarrow \infty} \sin x$  does not exist. An analytical way to approach this problem is to apply the Squeeze Theorem:

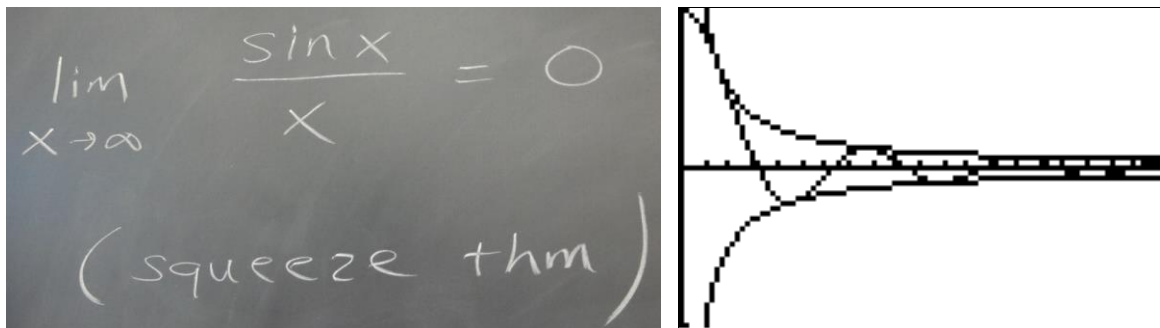
$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$  (table/graph)

$\lim_{x \rightarrow \infty} \sin x$  DNE

Start  $-1 \leq \sin x \leq 1$

$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$

let  $x \rightarrow \infty$



Notice how the graph of  $y = \frac{\sin x}{x}$  along with  $y = \frac{1}{x}$  and  $y = -\frac{1}{x}$  further support this argument.

## PROBLEM

Determining  $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x})$ .....

- (a) What does your intuition tell you about  $x - \sqrt{x^2 + x}$  for large values of  $x$ ?
- (b) What does a table say?
- (c) Find  $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x})$  analytically (this is challenging).

$\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x})$       Form:  $\infty - \infty = 0$

(A) guess ( $L=0$ )

(B) TI-84:  $L = -\frac{1}{2}$

(C) analytically

$$\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + x}) \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}}$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + x)}{x + \sqrt{x^2 + x}}$$

$\lim_{x \rightarrow \infty} \frac{-x}{x + \sqrt{x^2 + x}}$        $x \rightarrow \infty$

$= \lim_{x \rightarrow \infty} \frac{-x}{x + x\sqrt{1 + \frac{1}{x}}}$        $\div x$

$= \lim_{x \rightarrow \infty} \frac{-1}{1 + \sqrt{1 + \frac{1}{x}}}$       let  $x \rightarrow \infty$

$= \frac{-1}{1 + \sqrt{1+0}} = -\frac{1}{2}$       **Answer**

$\sqrt{x^2 + x} = \sqrt{x^2(1 + \frac{1}{x})}$   
 $= \sqrt{x^2} \sqrt{1 + \frac{1}{x}}$   
 $= x \sqrt{1 + \frac{1}{x}}$

↑

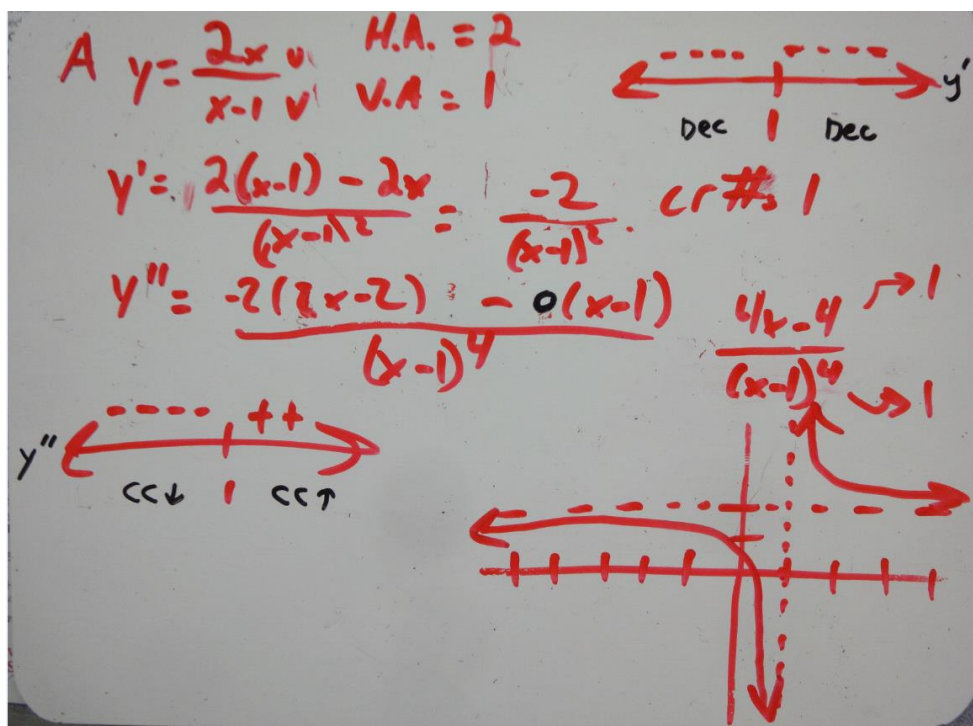
side work

### 3.5: A Summary of Curve Sketching (Review of previous sections)

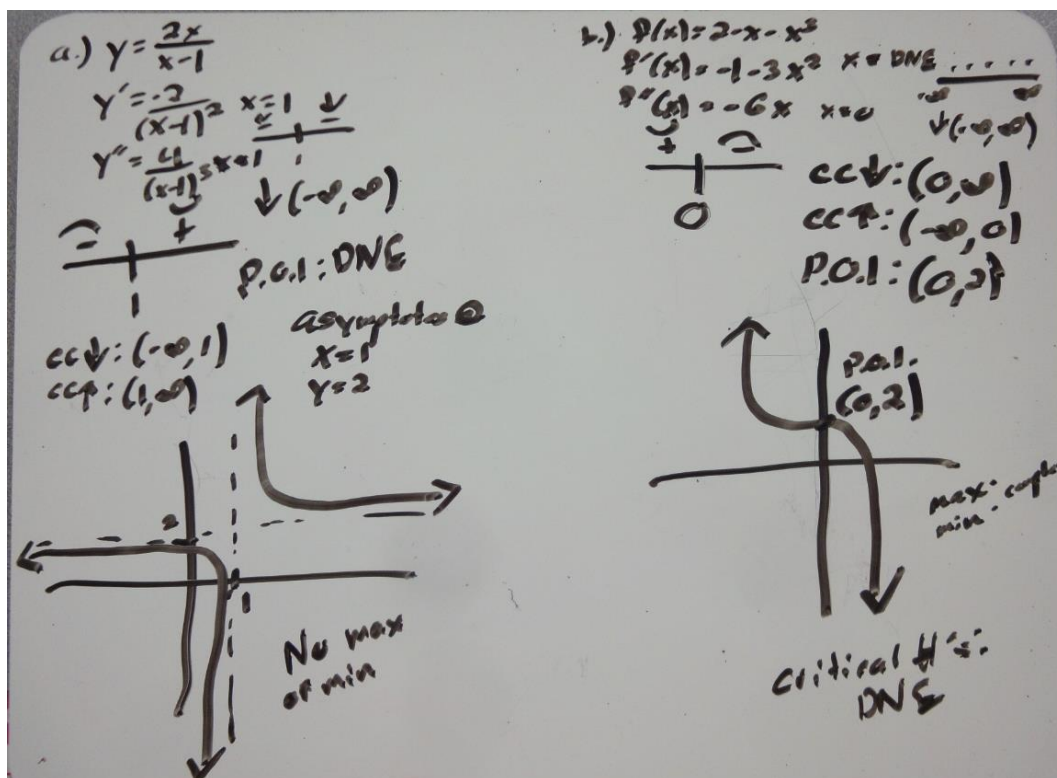
#### PROBLEM

Carry out a **full analysis** of the function.  
 In other words, study the **first derivative** (intervals of increase/decrease, extreme values), the **second derivative** (concavity, points of inflection), and any other pertinent characteristics (asymptotes, cusps, etc.). Also, provide a freehand sketch based on your findings.

(a)  $y = \frac{2x}{x-1}$       (b)  $f(x) = 2 - x - x^3$



Problem A: If we were being careful here, we would not call  $x = 1$  a critical number. Why? It is because although it makes the derivative undefined, it is not in the domain of the function (so it couldn't possibly be a minimum or maximum). Even so, it's a good idea to include this number on the number line analysis for  $y'$ . Also, although the graph changes concavity at  $x = 1$ , this does not imply a point of inflection is there. Why? Because (again)  $x = 1$  is not in the domain of the function ( $x = 1$  is a vertical asymptote) so it could not correspond to an inflection point.



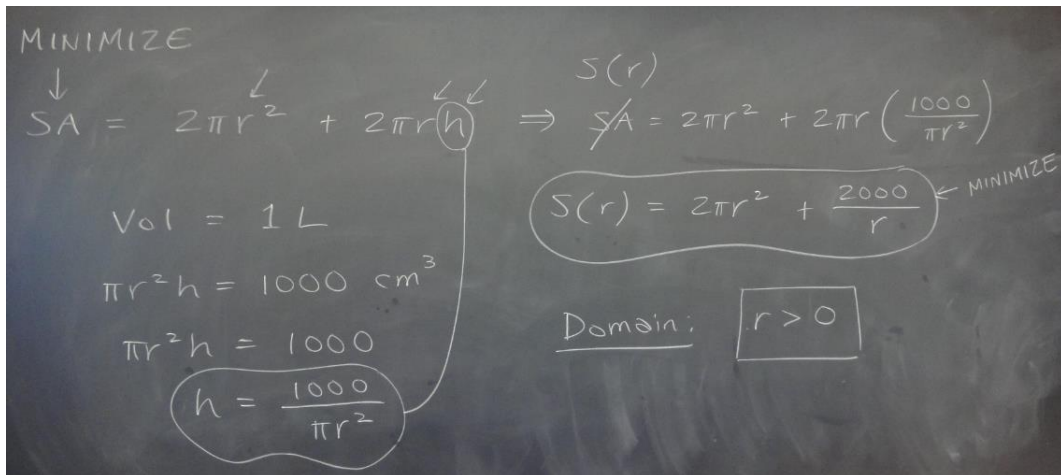
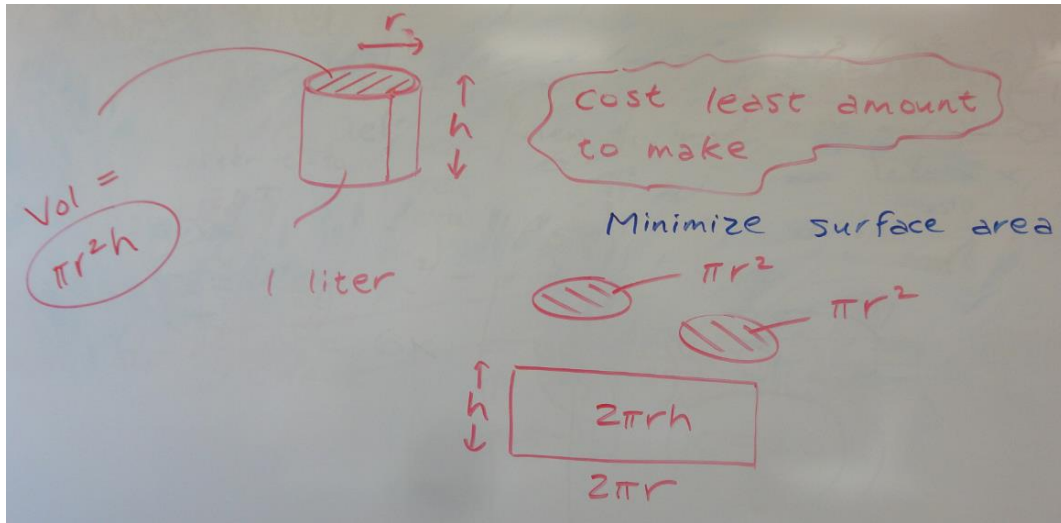
Problem B: An interesting feature here is that this function has no critical numbers. Any number tested in the derivative will be negative, implying that the function is decreasing from  $(-\infty, \infty)$ .

### 3.7: Optimization

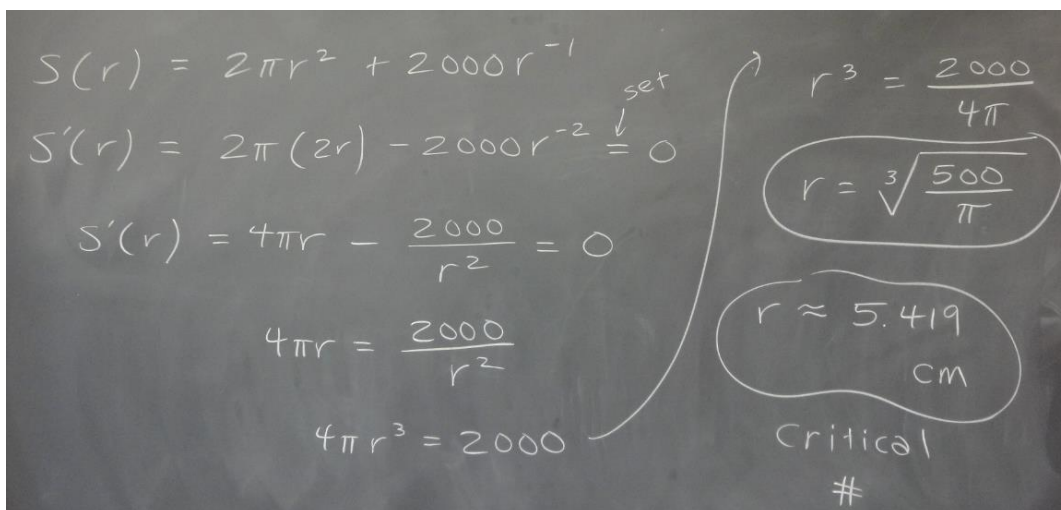
## PROBLEM

**A cylindrical can holds 1 liter of oil. What are the dimensions of the can that costs the least to manufacture?**

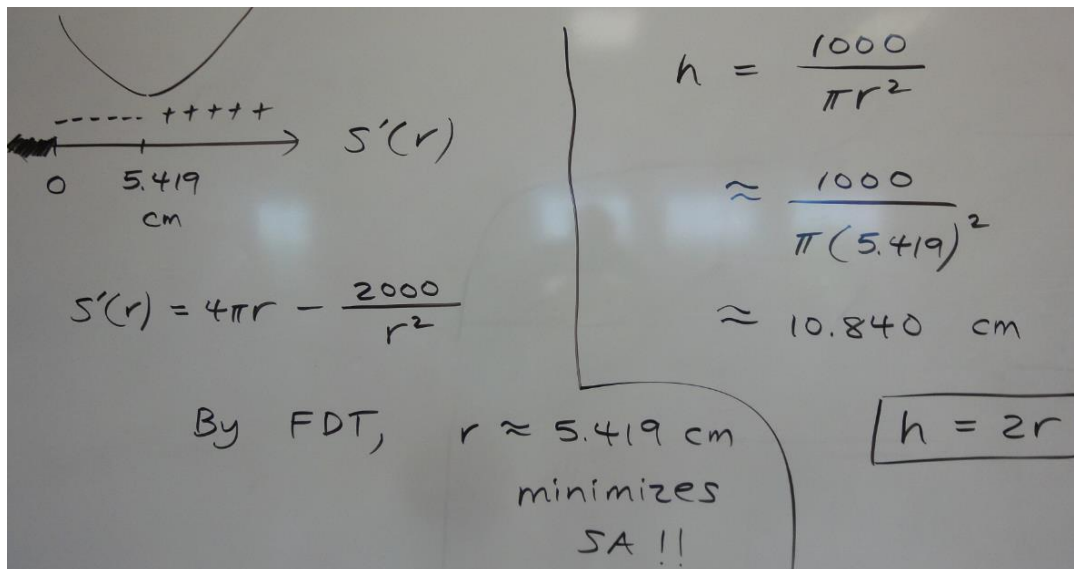




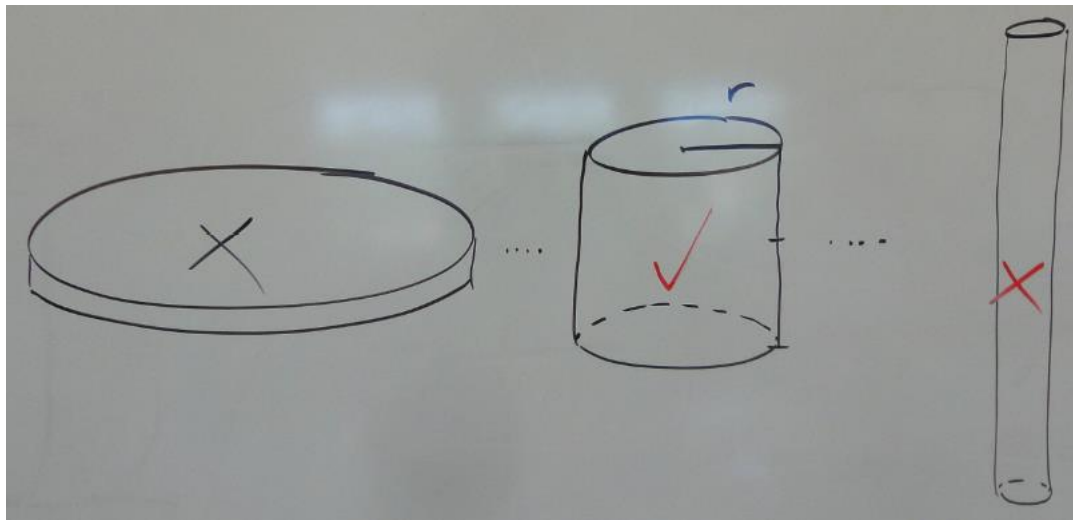
All of the math modeling is done above. We have the equation we need to optimize in terms of **one** variable ( $r$ ).



This is the step where Calculus (the derivative) helps.



Confirming that  $r \approx 5.419$  cm really corresponds to a minimum.

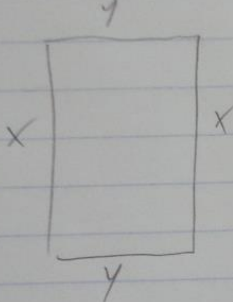


### PROBLEM (LEVEL I)

Find the length and width of a rectangle that has area of 75 square feet and a **MINIMUM** perimeter.

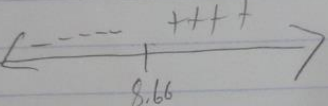


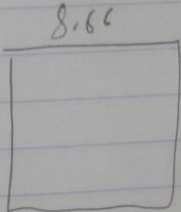
**Problem Level I** rectangle w/ 75 sq ft, minimum perimeter



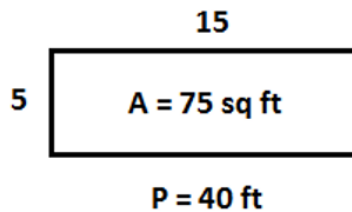
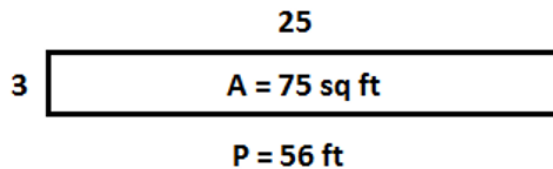
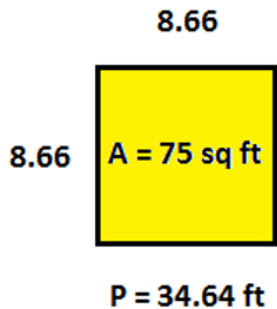
$SA = 2x + 2y$   
 $A: xy = 75 \quad y = \frac{75}{x}$   
 $f(x) = 2x + 2\left(\frac{75}{x}\right) \rightarrow 2x + \frac{150}{x}$   
 $f'(x) = 2 - \frac{150}{x^2} = 0 \quad 2 = \frac{150}{x^2}$   
 $2x^2 = 150$   
 $x^2 = 75$   
 $x = 8.66, y = 8.66$

$f'(8) = -.37$   
 $f'(9) = .1981$


  
 $x = 8.66$



## VISUAL SOLUTION



The optimal solution is seen in yellow (it's a square!). The other examples show a rectangle with area 75 sq ft but with perimeters larger than 34.64 ft.