

June 8, 2017

3.9: Antiderivatives (continued)

PROBLEM

Find the unique function

$f(x)$ such that

$$f'(x) = x^2 - 2x \text{ and } f(1) = \frac{1}{3}.$$

Handwritten solution for the antiderivative problem:

$$f'(x) = x^2 - 2x \quad ; \quad f(1) = \frac{1}{3}$$
$$f(x) = \frac{1}{3}x^3 - x^2 + C$$
$$\frac{1}{3} = \frac{1}{3}(1)^3 - (1)^2 + C \quad ; \quad \frac{1}{3} = -1 + \frac{1}{3} + C$$
$$C = 1$$

So... $f(x) = \frac{1}{3}x^3 - x^2 + 1$

PROBLEM

Suppose a stone is dropped from a cliff. The stone hits the water at 120 feet/sec. What is the height of the cliff?



Source: jonas.ph

terminal velocity = -120 ft/sec

$$a = -32 \text{ ft/s}^2$$

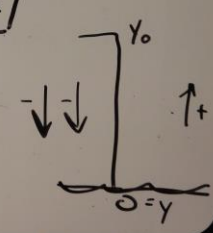
$$s(t) \longrightarrow v(t) \longrightarrow a(t)$$

Nabb's solution:

$a = -32$ stone dropped
 $v(t) = -32t + c$
 $v(0) = -32(0) + c = 0$
 $c = 0$
 $v(t) = -32t$ ✓
 $-120 = -32t \Rightarrow t = 3.75$ sec
 $s(t) = -16t^2 + d$
 $s(0) = -16(0)^2 + d$
 $s(0) = d$ ← height of cliff!
 $s(t) = -16t^2 + d$
 $0 = -16(3.75)^2 + d$
 $d = 225$ ft

Another nice solution:

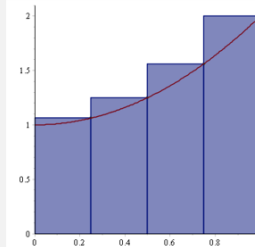
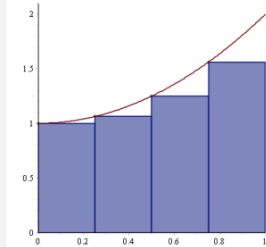
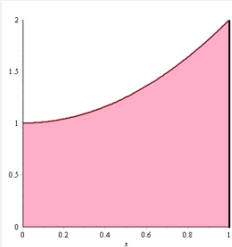
$a = -32 \text{ ft/s}^2$
 $v = -32t + v_0 \Rightarrow v_0 = 0$
 $y = -16t^2 + y_0$
 $-120 = -32t$
 $t = 3.75$
 $0 = -16(3.75)^2 + y_0$
 $y_0 = 225$ ft



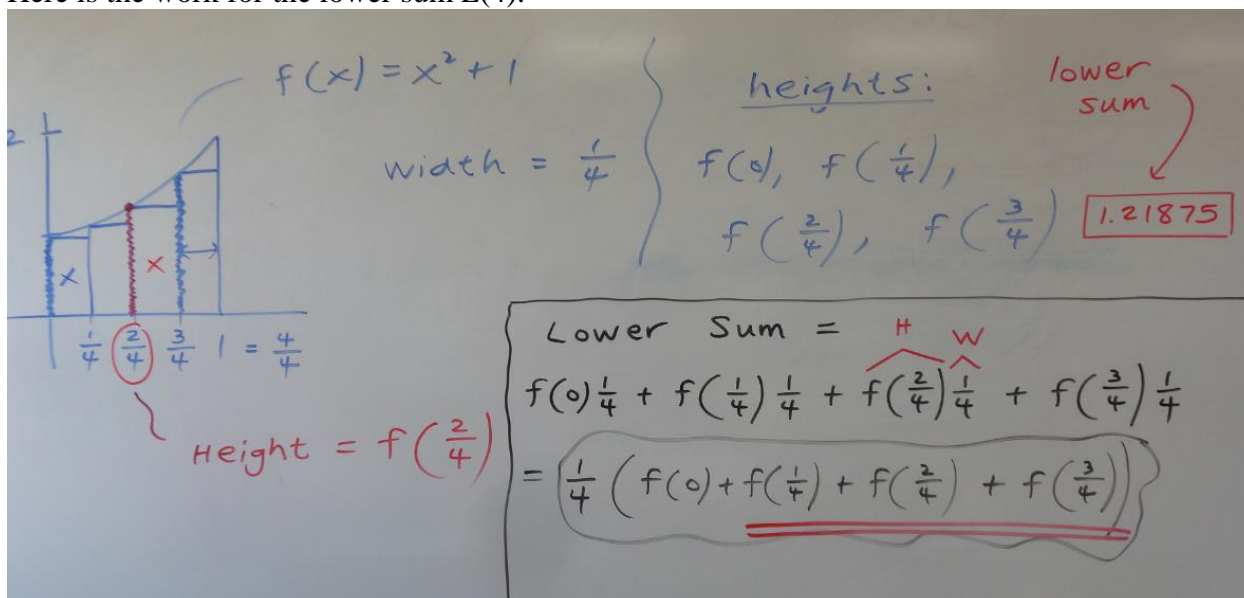
4.1: Area & Distance (Introduction)

PROBLEM

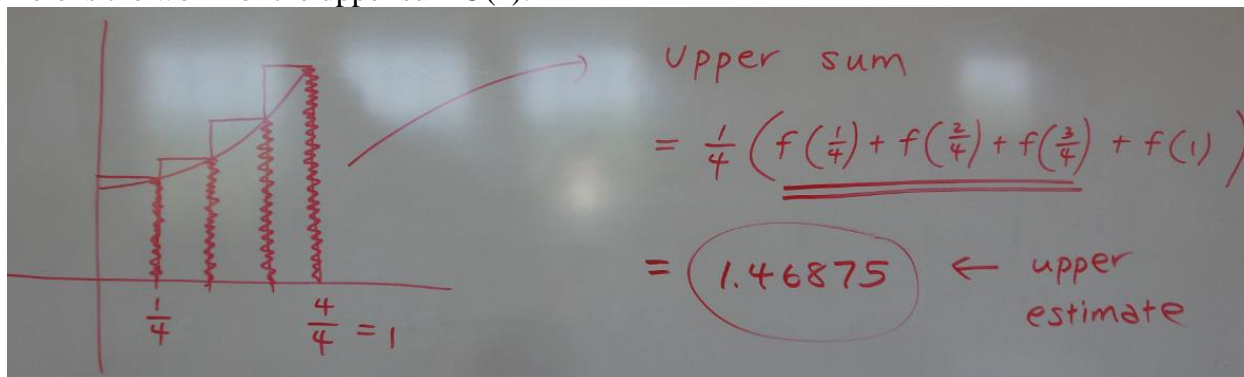
Consider the region bounded by the parabola $f(x) = x^2 + 1$, the x -axis, and the vertical lines $x = 0$ and $x = 1$. Let's estimate this area using simple objects: rectangles.



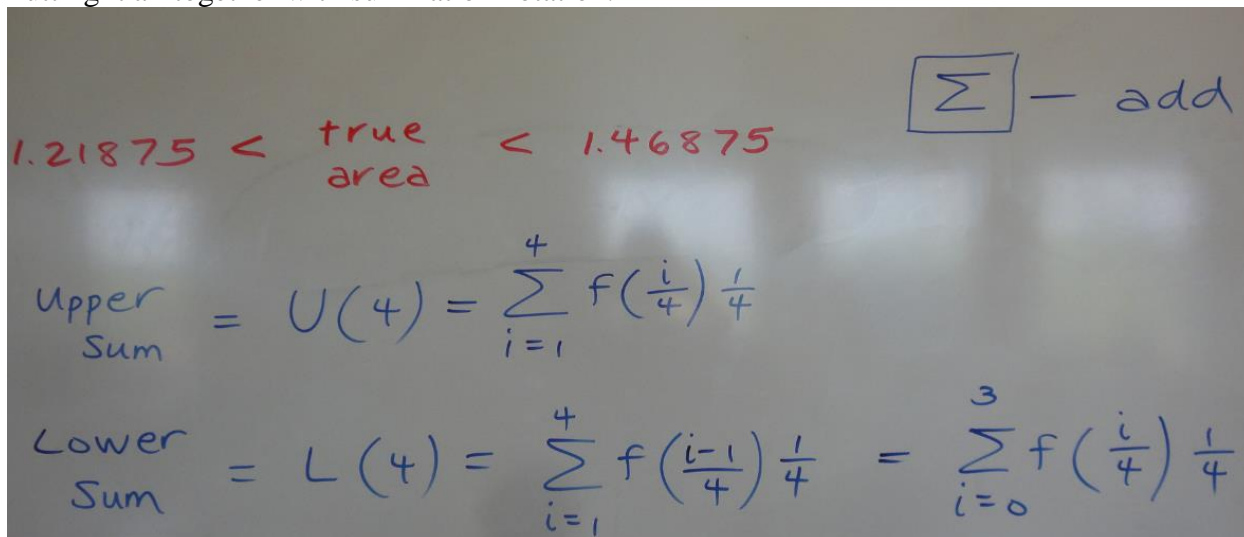
Here is the work for the lower sum $L(4)$:



Here is the work for the upper sum $U(4)$:



Putting it all together with summation notation:



As we can see in the table below, the lower sum continues to *increase* as the number of rectangles increases. The upper sum continues to *decrease* as the number of rectangles increases. Both are tending toward the same number (the exact area under the curve).

OBSERVATION

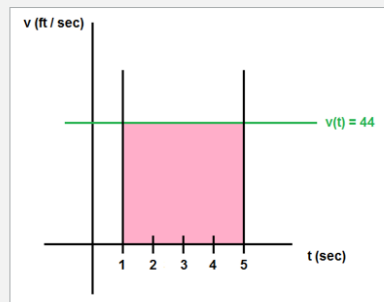
Approximate areas (both underestimates and overestimates) using n rectangles.

n	Lower Sum	Upper Sum
4	1.2188	1.4688
8	1.2734	1.3984
20	1.3088	1.3588
50	1.3234	1.3434
100	1.3284	1.3384
500	1.3323	1.3343

<http://www.shodor.org/interactivate/activities/Integrate/>

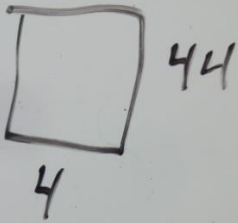
QUESTION

- (a) Consider a car moving along a straight road with constant velocity $v(t) = 44$ ft/sec. What is the distance traveled by the car from $t = 1$ sec to $t = 5$ sec?
- (b) What geometric calculation gives the answer (instantly!) to the above question? See the figure.



a) $4 \times 44 = 176 \text{ ft}$

B)



$176 = w \cdot h =$
 $xy = 176$

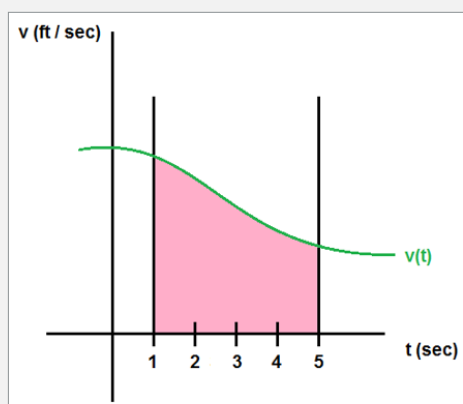
$d = vt$

$44 \text{ ft/sec} \cdot 4 \text{ sec} = d$
 $d = 176 \text{ ft}$

In this context, displacement/distance is equivalent to the area under the curve.

FOLLOW UP

Just a quick question: Would the same reasoning apply to a car with variable (nonconstant) velocity?



ABSOLUTELY!!!

4.2: The Definite Integral

SUMMATION FORMULAS (PAGE 309)

- $$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$
- $$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

Playing around with summation:

Handwritten calculations showing the summation of the first 6 integers and the first 5 squares. The first sum is $\sum_{i=1}^6 i = 1+2+3+4+5+6 = 21 = \frac{6(6+1)}{2}$. The second sum is $\sum_{i=1}^5 i^2 = 1^2+2^2+3^2+4^2+5^2 = 55 = \frac{5(5+1)(2 \cdot 5 + 1)}{6}$.

EXERCISE

Can you prove

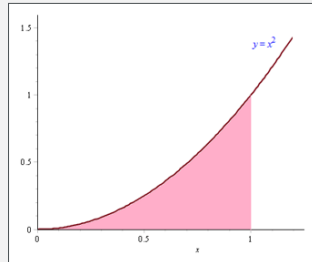
that
$$\sum_{i=1}^n i = \frac{n(n+1)}{2}?$$

Outline of the proof:

Handwritten outline of the proof for the summation formula. It shows the formula $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and then the sum written as $1+2+3+\dots+(n-2)+(n-1)+n$. The sum is then written in reverse order: $n+(n-1)+(n-2)+\dots+3+2+1$. The two sums are added together, and the result is $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

PROBLEM

Consider the region bounded by $f(x) = x^2$, $x = 0$, $x = 1$, and the x -axis:



Use an infinite number of rectangles to determine this area **exactly**.

$f(x) = x^2$
 $x = 0 \rightarrow 1$

Find exact area under curve
 (# rectangles = ∞)

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \overbrace{f(x_i)}^H \overbrace{\Delta x}^W$$

\uparrow ADD

rectangles = n

Upper Sum = $U(n)$

width = $\Delta x = \left(\frac{1}{n}\right)$

heights used: $f\left(\frac{1}{n}\right), f\left(\frac{2}{n}\right), \dots, f\left(\frac{n}{n}\right)$

$$U(n) = \sum_{i=1}^n f\left(\frac{i}{n}\right) \frac{1}{n}$$

$$= \sum_{i=1}^n \left(\frac{i}{n}\right)^2 \frac{1}{n}$$

$$U(n) = \sum_{i=1}^n \frac{i^2}{n^2} \cdot \frac{1}{n}$$

$$= \frac{1}{n^3} \sum_{i=1}^n i^2$$

$$\frac{n(n+1)(2n+1)}{6}$$

$$U(n) = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$U(n) = \frac{2n^2 + 3n + 1}{6n^2}$$

$U(10)$
 $U(100)$

$$U(n) = \frac{2n^2 + 3n + 1}{6n^2}$$

let $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} U(n) = \lim_{n \rightarrow \infty} \frac{2n^2 + 3n + 1}{6n^2} = \frac{2}{6} = \frac{1}{3}$$

Exact Area

$\frac{1}{3}$ units²

$$\int_0^1 x^2 dx = \frac{1}{3}$$

Doing this on the calculator:

$$\int_a^b f(x) dx = \text{fnInt}(f(x), x, a, b)$$

$$\int_0^1 (x^2) dx$$

.3333333333

or

