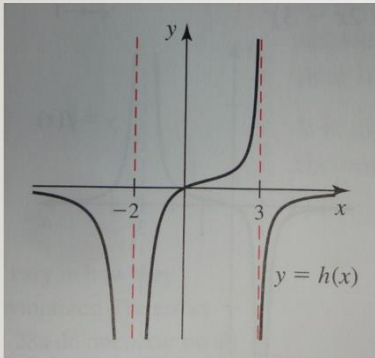


May 23, 2017

### 1.5B: Infinite Limits/Vertical Asymptotes

#### PROBLEM



Source: Briggs, Cochran, Gillett, 2016

Calculate the following, if possible:

- |                                      |                                     |
|--------------------------------------|-------------------------------------|
| (a) $\lim_{x \rightarrow -2^-} h(x)$ | (d) $\lim_{x \rightarrow 3^-} h(x)$ |
| (b) $\lim_{x \rightarrow -2^+} h(x)$ | (e) $\lim_{x \rightarrow 3^+} h(x)$ |
| (c) $\lim_{x \rightarrow -2} h(x)$   | (f) $\lim_{x \rightarrow 3} h(x)$   |

Handwritten solutions for the limit problems:

$$\lim_{x \rightarrow -2^-} h(x) = -\infty \quad \lim_{x \rightarrow 3^-} h(x) = \infty$$
$$\lim_{x \rightarrow -2^+} h(x) = \infty \quad \lim_{x \rightarrow 3^+} h(x) = -\infty$$
$$\lim_{x \rightarrow -2} h(x) = -\infty \quad \lim_{x \rightarrow 3} h(x) = \text{DNE}$$

VA  $x = -2, x = 3$

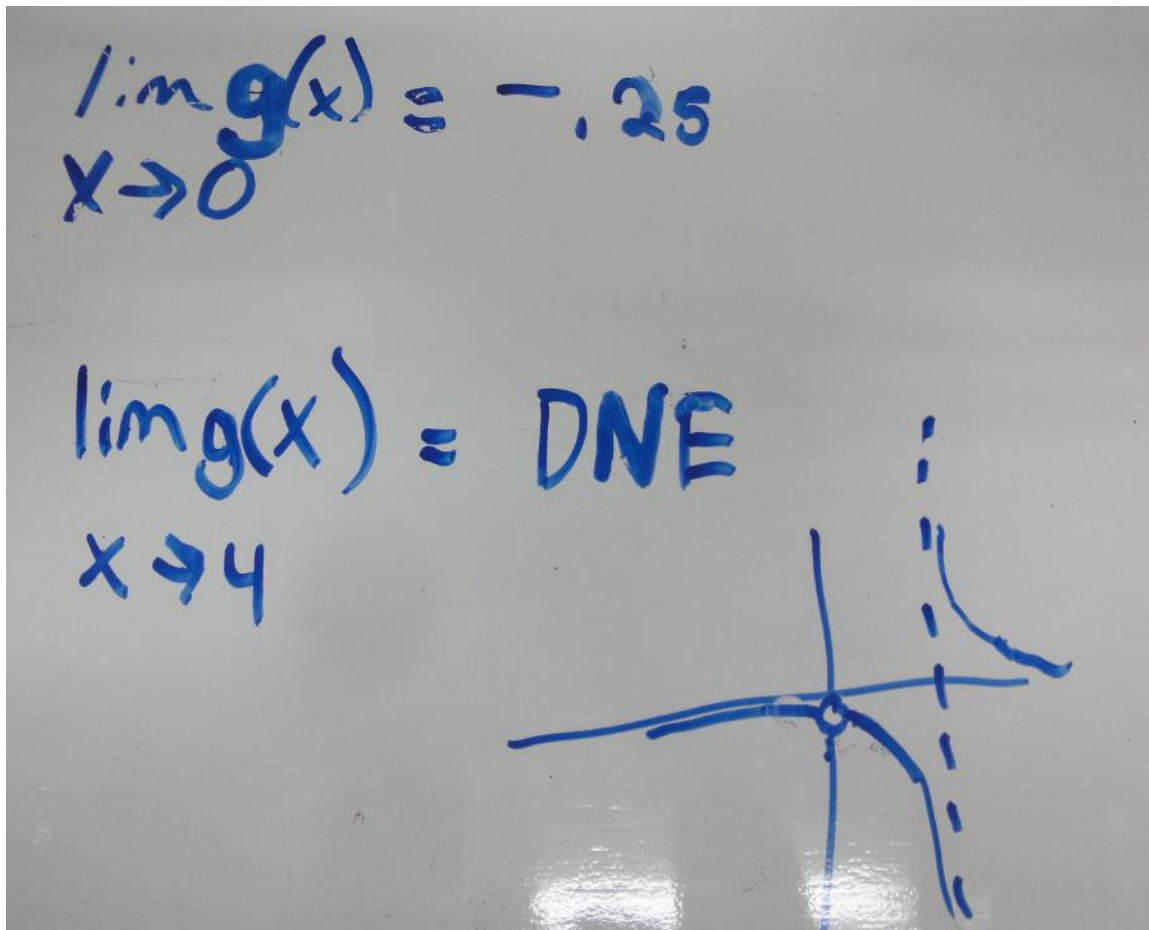
All of the above could be answered with a “does not exist” but using  $\infty$  and  $-\infty$  is preferred since it is more descriptive.

## PROBLEM

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Consider the function  $g(x) = \frac{x}{x^2 - 4x}$ . Examine

$\lim_{x \rightarrow 0} g(x)$  and  $\lim_{x \rightarrow 4} g(x)$  and describe what is happening in each case.



Near  $x = 0$ : The function tends to  $-0.25$  on either side of zero (notice the hole/discontinuity in the graph at  $(0, -1/4)$ ).

Near  $x = 4$ : The graph shows unbounded behavior so there is a vertical asymptote at  $x = 4$ .

## PROBLEM

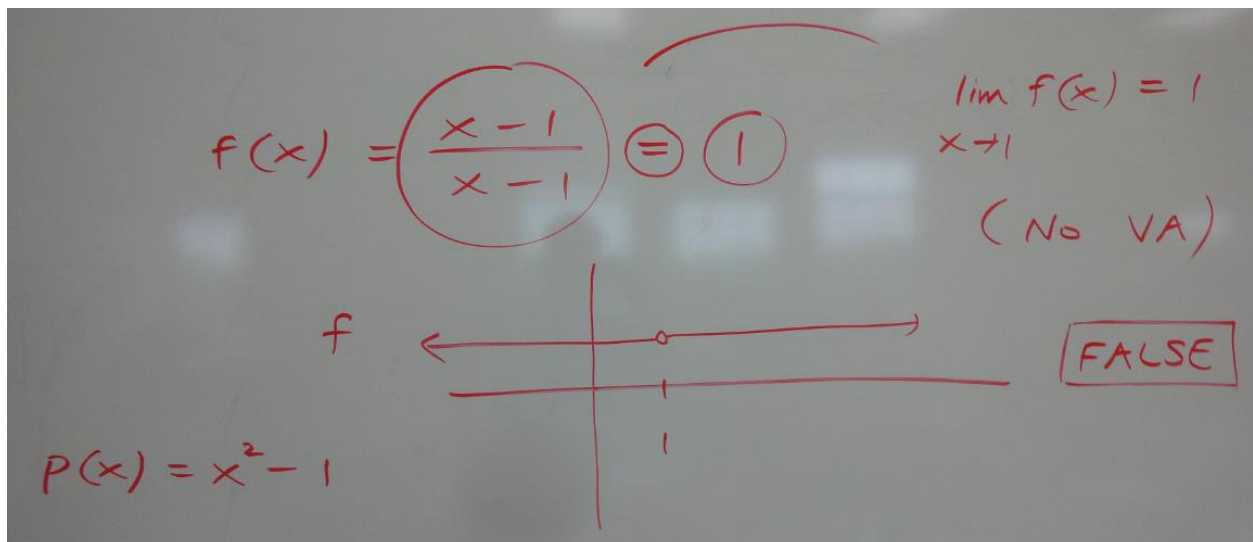
### TRUE / FALSE :

If  $p(x)$  is a polynomial, then the function

$f(x) = \frac{p(x)}{x-1}$  has a vertical asymptote at  $x=1$ .

### JUSTIFY!!

This is false!!! If you pull a  $p(x)$  out of thin air,  $f(x) = \frac{p(x)}{x-1}$  will *most likely* have a vertical asymptote at  $x=1$  (so  $\lim_{x \rightarrow 1} f(x)$  will not exist). However, there are many choices for  $p(x)$  where  $\lim_{x \rightarrow 1} f(x)$  will exist. See picture below, with  $p(x) = x-1$ :



In fact, any  $p(x)$  for which  $x-1$  is a factor is enough to show that this statement is false.

## 1.6A: Limit Laws (Finding Limits Analytically)

### WARM UP

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### TRUE / FALSE

Given  $f(x) = \frac{x^2 - 4}{x - 2}$  and  $g(x) = x + 2$ , we can say the functions  $f$  and  $g$  are equal. Explain your reasoning!

This board sums it up well. In order for two functions to be equal, they must have identical outputs for each input.

False

$$f(2) = \frac{2^2 - 4}{2 - 2} = \frac{0}{0} \text{ undefined}$$
$$g(2) = 2 + 2 = 4$$

@  $x=2$ ,  $f(x) \neq g(x)$

## PROBLEM

Attempt paper/pencil methods to determine these limits.

$$(a) \lim_{t \rightarrow 1} \frac{t^2 + t + 2}{t + 1} \quad (c) \lim_{r \rightarrow 0} \frac{\sqrt{r+1} - 1}{r}$$

$$(b) \lim_{x \rightarrow -3} \frac{x^2 + x - 6}{x + 3} \quad (d) \lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x}$$

(a) Direct substitution works here:

Handwritten solution for (a):

$$\lim_{t \rightarrow 1} \frac{t^2 + t + 2}{t + 1} = \frac{1^2 + 1 + 2}{1 + 1} = \frac{4}{2} = 2$$

(b) Factor/cancel to determine limit:

Handwritten solution for (b):

$$\lim_{x \rightarrow -3} \frac{(x-2)(x+3)}{x+3} = \lim_{x \rightarrow -3} (x-2) = -3 - 2 = -5$$

(c) Rationalize/Use the conjugate:

Handwritten solution for (c):

$$\lim_{r \rightarrow 0} \frac{\sqrt{r+1} - 1}{r} \cdot \frac{\sqrt{r+1} + 1}{\sqrt{r+1} + 1} = \lim_{r \rightarrow 0} \frac{(r+1) - 1}{r(\sqrt{r+1} + 1)} = \lim_{r \rightarrow 0} \frac{r}{r(\sqrt{r+1} + 1)} = \lim_{r \rightarrow 0} \frac{1}{\sqrt{r+1} + 1} = \frac{1}{\sqrt{0+1} + 1} = \frac{1}{2}$$

(d) Manipulate algebraically:

$$\begin{aligned} \lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x} &= \lim_{x \rightarrow -4} \frac{\frac{1}{4} \frac{x}{x} + \frac{1}{x} \frac{4}{4}}{4 + x} \\ &= \lim_{x \rightarrow -4} \frac{\frac{x+4}{4x}}{4+x} \\ &= \lim_{x \rightarrow -4} \frac{x+4}{4x} \cdot \frac{1}{4+x} \\ &= \lim_{x \rightarrow -4} \frac{1}{4x} \\ &= \frac{1}{4 \cdot (-4)} \\ &= \boxed{-\frac{1}{16}} \end{aligned}$$

**IMPORTANT:** Problems (b)-(d) all begin with the 0/0 form so they need some kind of manipulation (factoring, multiplying by a conjugate, getting common denominators, rewriting, etc.) to reveal the limit. Note that each of these can be checked on a calculator by using the TABLE feature.

## PROBLEM

Consider the following:  $\lim_{x \rightarrow 0} \frac{|2x-1| - |2x+1|}{x}$

- (a) Explain why evaluation of this limit is not straightforward.
- (b) Find the limit by any means. (Try to find the limit analytically.)

A graphing calculator will tell us  $L = -4$  (table or graph). To solve this problem analytically, use what we know about absolute value. That is,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$|2x-1| = \begin{cases} 2x-1, & 2x-1 \geq 0 \rightarrow 2x \geq 1, x \geq \frac{1}{2} \\ -(2x-1), & 2x-1 < 0 \rightarrow x < \frac{1}{2} \end{cases}$$
$$|2x+1| = \begin{cases} 2x+1, & 2x+1 \geq 0 \rightarrow x \geq -\frac{1}{2} \\ -(2x+1), & 2x+1 < 0 \rightarrow x < -\frac{1}{2} \end{cases}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{|2x-1| - |2x+1|}{x} &= \lim_{x \rightarrow 0} \frac{-(2x-1) - (2x+1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{-2x + 1 - 2x - 1}{x} \\ &= \lim_{x \rightarrow 0} \frac{-4x}{x} \\ &= \lim_{x \rightarrow 0} (-4) \\ &= -4 \end{aligned}$$

for  $x \approx 0$



## Lesson 1.6B: The Squeeze Theorem

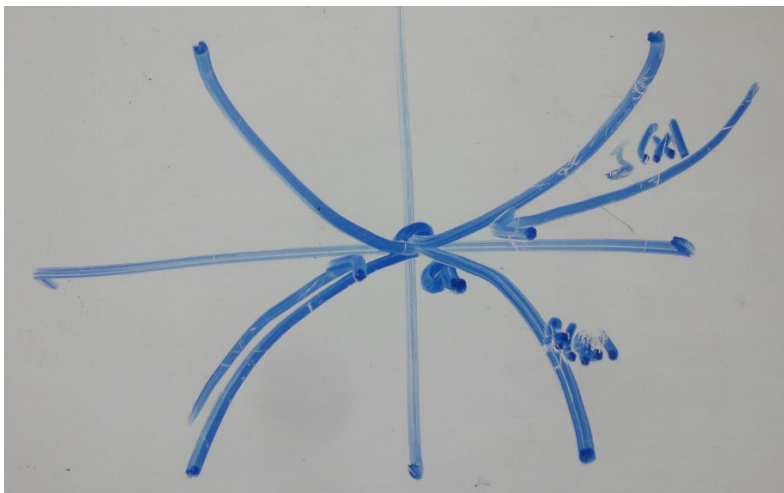
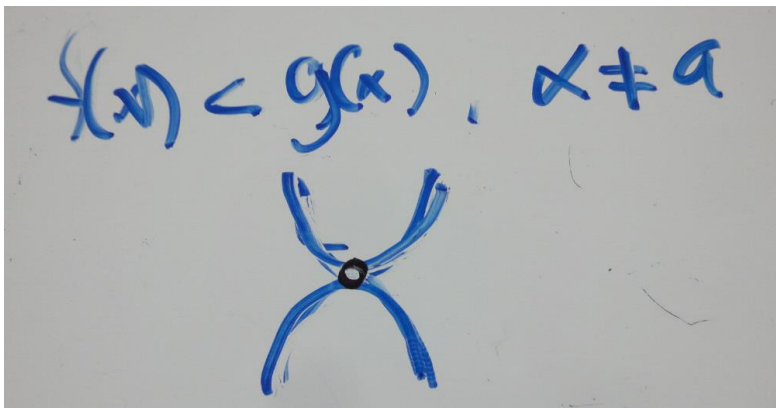
### PROBLEM

### TRUE / FALSE

If  $f(x) < g(x)$  for all  $x \neq a$ ,  
then  $\lim_{x \rightarrow a} f(x) < \lim_{x \rightarrow a} g(x)$ .

Justify!!

These two boards capture the same idea (it is FALSE). It could be that  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$ :



## PROBLEM

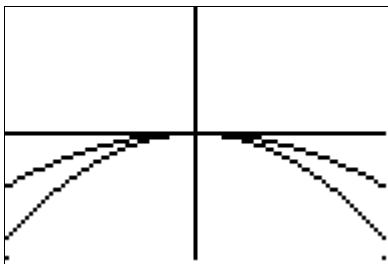
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Graph the three functions

$$f_1(x) = 1 - \frac{x^2}{6}, \quad f_2(x) = \frac{x \sin x}{2 - 2 \cos x}, \quad f_3(x) = 1$$

on your calculator. Can you determine the value of  $\lim_{x \rightarrow 0} f_2(x)$ ?

The quickest route here is to graph the functions and study what happens as  $x \rightarrow 0$ . The window below is  $[-1, 1, 0.8, 1.2]$ .



From the graph, we can see that  $f_2$  is squeezed between  $f_1$  and  $f_3$  (each approaches 1 for  $x \rightarrow 0$ ). Based on this,  $\lim_{x \rightarrow 0} f_2(x) = 1$ .