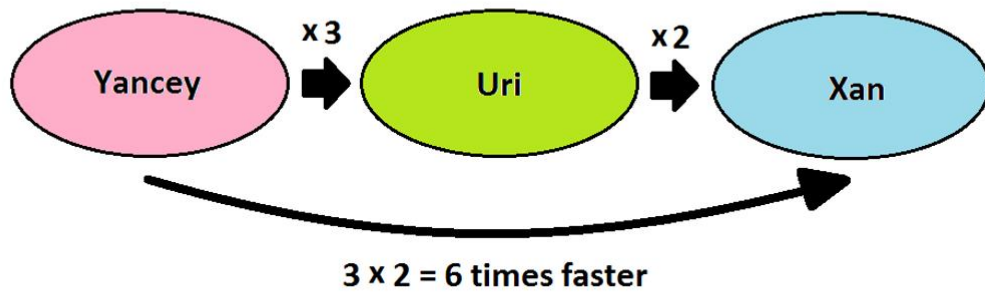


May 30, 2017

## 2.5: The Chain Rule

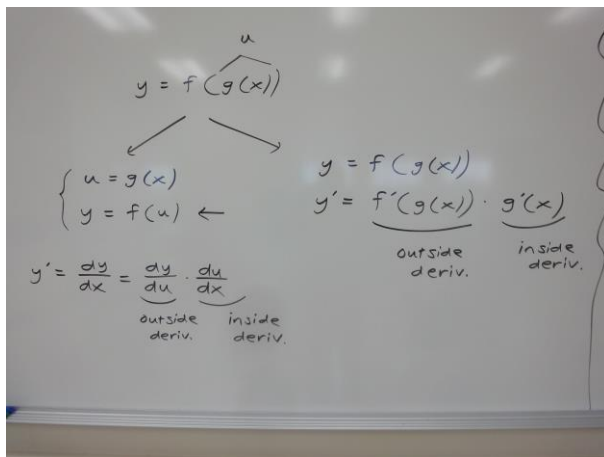
### APPLE PICKING



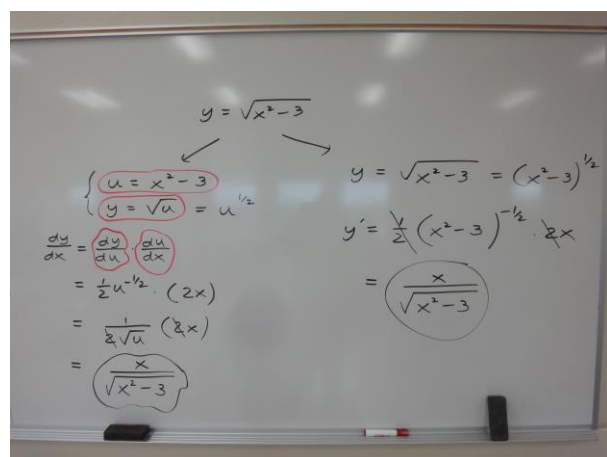
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6$$

Source: Briggs, Cochran & Gillett, 2015

Two Perspectives:



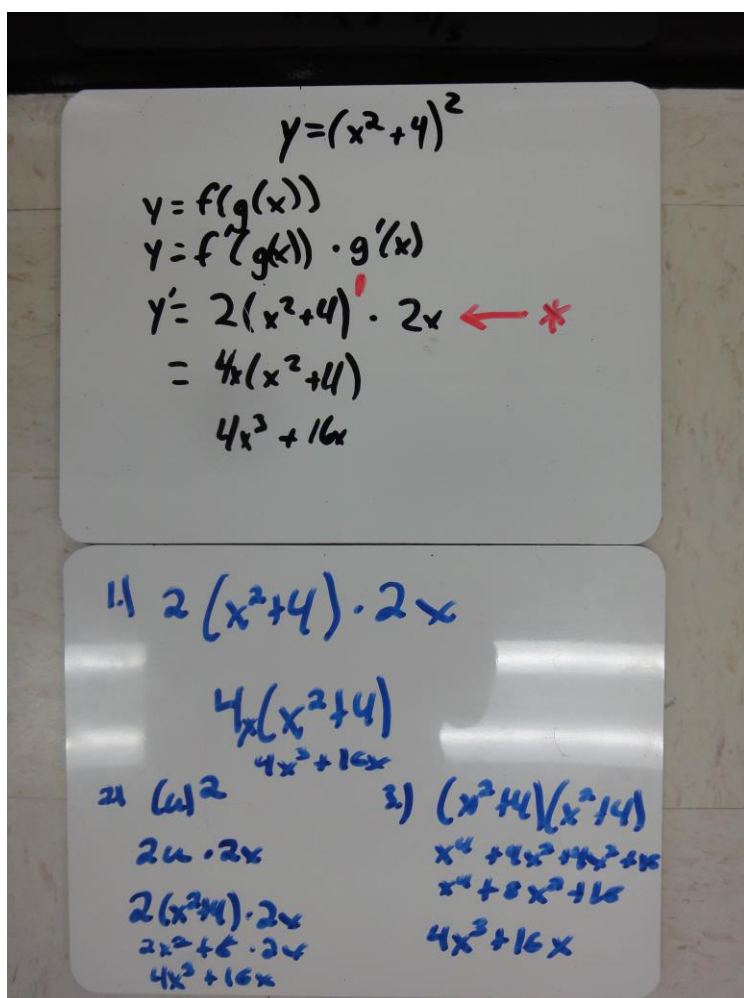
Sample Problem:



## WARM UP

Consider the function  $y = (x^2 + 4)^2$ .

You now have two different ways to differentiate this function. Show that the final answers agree.



Also worth mentioning: You can use the product rule on  $y = (x^2 + 4)(x^2 + 4)$  so

$$y' = (x^2 + 4)(2x) + (2x)(x^2 + 4) = 2(2x)(x^2 + 4) = \boxed{4x^3 + 16x}.$$

## PROBLEM (LEVEL I)

Find the derivative.

(a)  $y = (7x + 4)^4$

(c)  $y = \sec(x^2 + 3)$

(b)  $y = \sqrt{5x - 2}$

(d)  $y = \sqrt{x + \sqrt{x}}$

(A)  $y = (7x + 4)^4$   
 $\begin{cases} u = 7x + 4 \\ y = u^4 \end{cases}$   
 $y' = 4u^3 \cdot 7$   
 $= 28(7x + 4)^3$

(B)  $y = \sqrt{5x - 2}$   
 $y' = \frac{1}{2}(5x - 2)^{-1/2} \cdot 5$   
 $y' = \frac{5}{2\sqrt{5x - 2}} (5x - 2)^{1/2}$

(c)  $y = \sec(x^2 + 3)$   
 $\sec(u)$   
 $\sec(u) \tan(u) \cdot 2x$   
 $y' = 2x \sec(x^2 + 3) \tan(x^2 + 3)$

(d)  $y = \sqrt{x + \sqrt{x}}$   
 $(x + x^{1/2})^{1/2}$   
 $\frac{1}{2}(x + x^{1/2})^{1/2} \cdot (1 + \frac{1}{2}x^{-1/2})$   
 $y' = \left(\frac{1}{2\sqrt{x + \sqrt{x}}}\right) \left(1 + \frac{1}{2\sqrt{x}}\right)$

## PROBLEM

Find the derivatives of

$$y = \sin(x^4) \quad \text{and} \quad y = \sin^4 x.$$

Explain the similarities and differences in technique.

LHS

$$y = \sin(x^4)$$

$$u = x^4$$

$$y = \sin(u)$$

RHS

$$y = \sin^4 x$$

$$= (\sin x)^4$$

$$u = \sin x$$

$$y = u^4$$

The above boards show that although the same two functions are at work here—the **sine** function and the **fourth power**—they play different roles in the two problems. In  $y = \sin(x^4)$ , the  $x^4$  is the inside function while sine is the outside function. This is reversed in the problem  $y = \sin^4 x = (\sin x)^4$ . See below:

$y = \sin(x^4)$ $y = \sin(u)$ $y' = \cos(u) \cdot 4x^3$ $y' = 4x^3 \cos(x^4)$	$y = \sin(x^4) \quad   \quad y = \sin^4(x)$ $y' = \cos(x^4) \cdot 4x^3 \quad   \quad y' = 4 \sin^3(x) \cos(x)$
$y = \sin(x^4)$ $u = x^4 \rightarrow 4x^3$ $y = \sin x \rightarrow \cos x$ $y = f(g(x)) \cdot q(x)$ $y' = f'(g(x)) \cdot g'(x)$ $y' = \cos(x^4) \cdot 4x^3$ $y' = 4x^3 \cos(x^4)$	$y = (\sin^4 x) \quad (u)^4$ $4(u)^3 \cdot \cos x$ $4(\sin^3 x) \cdot \cos x$ $4 \sin^3 x \cos x$

## PROBLEM (LEVEL II)

Find the derivative.

(a)  $y = \cos^3(2x)$

(c)  $y = x^3 \sqrt{1-x^2}$

(b)  $y = x \tan(7x)$

(d)  $y = \left( \frac{\sin x}{x+2} \right)^2$

$$y' = 3 (\cos 2x)^2 \cdot (-\sin 2x) \cdot 2$$

$$= -6 \cos^2 2x \sin 2x$$

Correct!

Notice **two** applications of the chain rule above!!

(b)  $y = x \cdot \tan(7x)$

$$(x) (\tan 7x) = (x \cdot \sec^2 7x \cdot 7) + (\tan 7x \cdot 1)$$

$$y' = 7x \sec^2 7x + \tan 7x$$

Product

Product & Chain needed above.

(c)  $y = x^3 \sqrt{1-x^2}$

NABB

$$y' = x^3 \cdot \frac{1}{2} (1-x^2)^{-1/2} \cdot (-2x) + \sqrt{1-x^2} \cdot (3x^2)$$

$$= \frac{-x^4}{\sqrt{1-x^2}} + 3x^2 \sqrt{1-x^2}$$

optional

$$\frac{-x^4}{\sqrt{1-x^2}} + 3x^2 \sqrt{1-x^2} = \frac{-x^4 + 3x^2(1-x^2)}{\sqrt{1-x^2}} = \frac{3x^2 - 4x^4}{\sqrt{1-x^2}}$$

Product & Chain needed above.

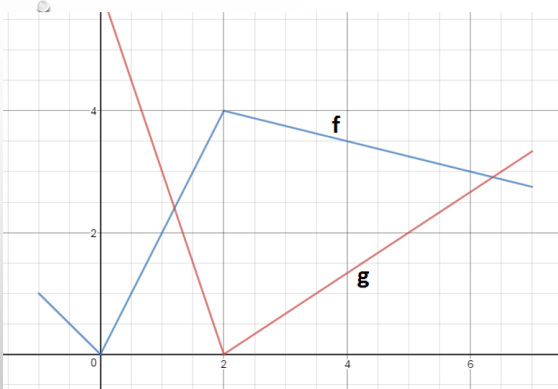
(d)  $y = \left( \frac{\sin x}{x+2} \right)^2$

$$y' = 2 \left( \frac{\sin x}{x+2} \right) \cdot \frac{(x+2)(\cos x) - (\sin x)(1)}{(x+2)^2}$$

$$y' = \frac{2 \sin x (x+2) \cos x - 2 \sin x}{(x+2)^2}$$

Power      quotient rule

Power & Quotient needed above.



If  $u(x) = f(g(x))$ ,  
 $v(x) = g(f(x))$ , and  
and  $w(x) = g(g(x))$ ,  
find the following,  
should they exist:

- $u'(1)$
- $v'(1)$
- $w'(1)$

find  
shou

- 
- 
- 

$$u'(1) = f'(g(1)) \cdot g'(1) \quad \text{CHAIN}$$

$$= f'(3) \cdot g'(1)$$

$$= \frac{-1/2}{2} \cdot \frac{-3}{1} = \left(\frac{3}{4}\right)$$

(B)  $v'(1) = g'(f(1)) \cdot f'(1)$   
 $= \underline{g'(2)} \cdot f'(1)$   
DNE

Does not exist

(C)  $w'(1) = g'(g(1)) \cdot g'(1)$   
 $= g'(3) \cdot \underline{g'(1)}$   
 $\frac{+1}{3/2} \cdot \frac{-3}{1}$   
 $m = \left(\frac{2}{3}\right)(-3)$   
 $= (-2)$

## 2.6: Implicit Differentiation

### WARM UP

Differentiate the following expressions with respect to  $x$ . Be sure to treat  $y$  as a **function** of  $x$ .

- (a)  $x^3$                       (c)  $x \sin y$   
(b)  $y^3$                       (d)  $\tan xy$

(a)  $\frac{d}{dx}(x^3) = 3x^2$

(b)  $y^3$   
 $\frac{d}{dx}(y^3) = ?$   
 $\frac{d}{dx}(y^3) = \frac{d}{dx}[y(x)]^3$   
 $= 3[y(x)]^2 \cdot y'(x)$   
 $= 3y^2 \frac{dy}{dx} = 3y^2 y'$

The big connection between (a) and (b) is that *the rule is the same* (Power Rule) but  $y$  is a function of  $x$  so we use the Chain Rule in part (b).

(c)  $\frac{d}{dx}(x \sin y)$   
 $= \frac{d}{dx}(x \cdot \sin y(x))$   
 $= x \cdot \cos y(x) \cdot y'(x) + \sin y(x) \cdot 1$   
 $= x y' \cos y + \sin y$

$= x \cdot \cos y(x) \cdot y'(x) + \sin y(x) \cdot 1$   
 $= x y' \cos y + \sin y$   
d)  $\tan xy$                        $x \rightarrow 1$   
 $= \frac{d}{dx}(\tan xy)$                        $y \rightarrow y'(x)$   
 $= \sec^2 xy ((1)(y) + y'(x)(x))$   
 $= \sec^2 xy (y + y'(x)(x))$   
 $= y \sec^2 xy + x \frac{dy}{dx} \sec^2 xy$

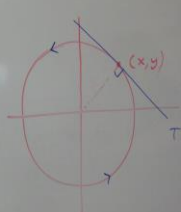
## PROBLEM

Consider the standard form for the unit circle  $x^2 + y^2 = 1$ . Find the first derivative  $\frac{dy}{dx}$  implicitly. What kind of geometric information does this provide?

$$\begin{aligned}x^2 + y^2 &= 1 \\ \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) \\ 2x + 2y \cdot \frac{dy}{dx} &= 0 \quad * \\ 2y \frac{dy}{dx} &= -2x \\ y \frac{dy}{dx} &= -x\end{aligned}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Slope of tangent line



$$\begin{aligned}m(\text{radial line}) &= \frac{y}{x} \\ \left(\frac{y}{x}\right)\left(-\frac{x}{y}\right) &= (-1)\end{aligned}$$

This problem illustrates that the slope of the tangent line at any point  $(x, y)$  on the circle is **perpendicular** to the radial line passing through  $(x, y)$ . This is something we can see in the diagram.