

May 31, 2017

Some Warming Up with the Chain Rule:

$$y = \sqrt{\cos 4x} = (\cos 4x)^{1/2}$$

power chain

$$y' = \frac{1}{2}(\cos 4x)^{-1/2}$$
$$\rightarrow \frac{1}{2\sqrt{\cos 4x}} \cdot \underbrace{-\sin 4x \cdot 4}$$
$$= \frac{\sin 4x \cdot 4}{2\sqrt{\cos 4x}} \quad \text{factor 2}$$
$$= \frac{\sin 4x \cdot 2}{\sqrt{\cos 4x}}$$
$$\frac{-2 \sin 4x}{\sqrt{\cos 4x}}$$

$$h(x) = 3x \sec^2 x$$
$$h(x) = (3x)(\sec x)^2$$
$$h'(x) = \underbrace{(3x)}_{=} \cdot \underbrace{2(\sec x)' \sec x \tan x}_{=} + \underbrace{(\sec^2 x)' \cdot 3}_{=}$$
$$= 3\sec^2 x (2x \tan x + 1)$$

2.6: Implicit Differentiation (continued)

PROBLEM

Consider the relation

$$x^2y + xy^2 = 1. \text{ Find } \frac{dy}{dx}$$

by implicit differentiation.

Three variations (all are correct):

Handwritten solution for the first variation of implicit differentiation:

$$x^2y + xy^2 = 1$$
$$\frac{dy}{dx} (2xy + y^2x^2 + x^2 + x2y y') = 0$$
$$y^2x^2 + x2y y' = -2xy - y^2$$
$$y'(x^2 + x2y) = -2xy - y^2$$
$$y' = -\frac{2xy + y^2}{x^2 + 2xy}$$

Handwritten solution for the second variation of implicit differentiation:

$$x^2y' + 2xy + x2y y' + y^2 = 0$$
$$2xy + y^2 = -x2y y' - x^2y'$$
$$\frac{2xy + y^2}{-x2y - x^2} = y'$$

Handwritten solution for the third variation of implicit differentiation:

5/31/17

$$x^2y + xy^2 = 1$$
$$\frac{d}{dx}(x^2y + xy^2) = \frac{d}{dx}(1)$$
$$(2x \frac{dy}{dx} + 2y \frac{dx}{dx}) = 0$$
$$x^2 = 2x \quad x \rightarrow 1$$
$$y = \frac{dy}{dx} \quad y^2 \rightarrow 2y \frac{dy}{dx}$$
$$(x^2 \frac{dy}{dx} + y(2x)) + (1(y^2) + 2y \frac{dy}{dx}(x)) = 0$$
$$x^2 \frac{dy}{dx} + 2yx + y^2 + 2xy \frac{dy}{dx} = 0$$
$$x^2 \frac{dy}{dx} + 2xy \frac{dy}{dx} + 2yx + y^2 = 0$$
$$\frac{1}{(x^2 + 2xy)} (x^2 \frac{dy}{dx} + 2xy \frac{dy}{dx}) = \frac{-2yx - y^2}{(x^2 + 2xy)}$$
$$\frac{dy}{dx} = \frac{-2yx - y^2}{x^2 + 2yx}$$

PROBLEM

Find $\frac{dy}{dx}$ implicitly for the equation $\sin y = x$. At some point in your work, make use of the trigonometric identity

$\sin^2 \theta + \cos^2 \theta = 1$ in order to express $\frac{dy}{dx}$ in terms of x .

You have just found the derivative of which important function?

$\sin y = x$
 $\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$
 $\cos y \cdot \frac{dy}{dx} = 1$
 $\frac{dy}{dx} = \frac{1}{\cos y}$
 $\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$
 $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (\sin y)^2}}$
 $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$
 $y = \sin^{-1} x$
 $y = \arcsin x$

From the above work, given $y = \sin^{-1} x = \arcsin x$, we get $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$.

2.7 : Applications of the Derivative

PROBLEM

In a fireworks display, a shell is launched vertically upward from the ground, reaching a height (in feet) of $s = -16t^2 + 256t$ after t sec. The shell is designed to burst when it reaches maximum altitude.

- When will the shell burst?
- What is the altitude of the shell the instant it explodes?

$s = -16t^2 + 256t$
 $s' = -32t + 256$
 $0 = -32t + 256$
 $-256 = -32t$
 $\frac{-256}{-32} = \frac{-32t}{-32}$
 $t = 8 \text{ sec}$
 $s = -16(8)^2 + 256(8) = 5$
 $s = -1024 + 2048$
 $s = 1024 \text{ ft}$

Important: We set $s'(t) = 0$ in part (a) because the shell bursts at maximum altitude. The shell momentarily stops (so velocity = 0).

PROBLEM

A machine is causing a particle to move along the x -axis so that its position at time t is given by

$$x(t) = (t-4)^2, \text{ where } t \text{ is in seconds.}$$

- (a) What is the particle's velocity at $t=2$? Interpret.
 (b) The machine stops suddenly at $t=3$, releasing the particle. As the particle continues, where will it be 5 seconds after the machine stops? Explain your thinking.

This shows the seeds of the formula $x = x_0 + vt$:

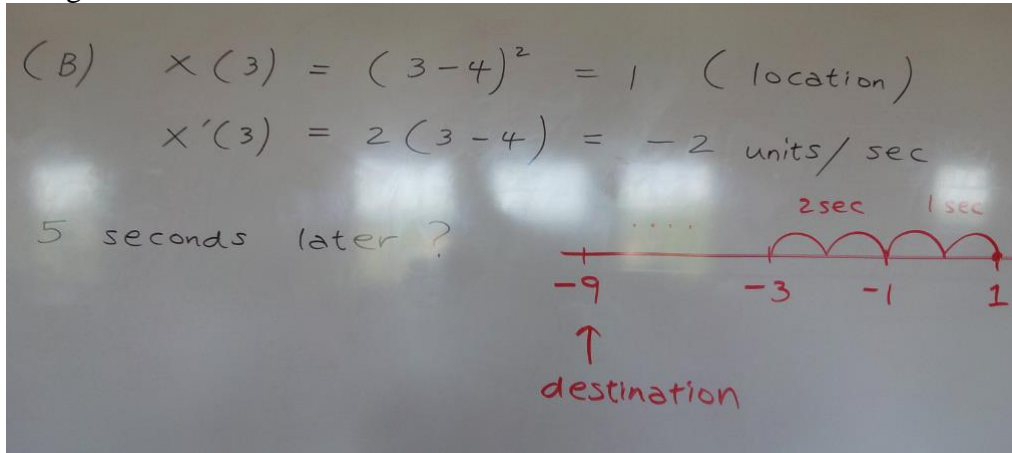
$x(t) = 2t - 8$ $x'(2) = -4 \text{ m/s}$

 $x'(3) = 2(3) - 8 = -2 \text{ m/s}$
 fixed velocity from $t=3$ and on.
 Position at $t=3 = (-1)^2 = 1$ | Distance traveled from $x=3 \rightarrow 8$
 $x=3$ | $= -2 \cdot 5 = -10 \text{ m/s}$
 Position at $t=8 = 1 + (-10) = -9$

Using a counting technique:

$x(t) = (t-4)^2$
 $x'(t) = 2(t-4) \cdot 1$ B. $x'(3) = 2(3-4) \cdot 1 = -2$
 $x'(2) = 2(2-4) \cdot 1 = -2$
 $A = -4$
 position
 $x(t) = (5-4)^2 = 1$
 $1 + -2 + -2 + -2 + -2 = -9$
 1sec 2sec 3sec 4sec 5sec

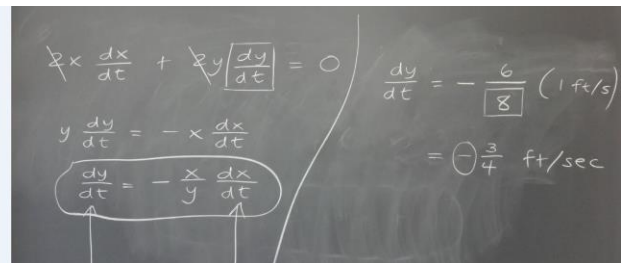
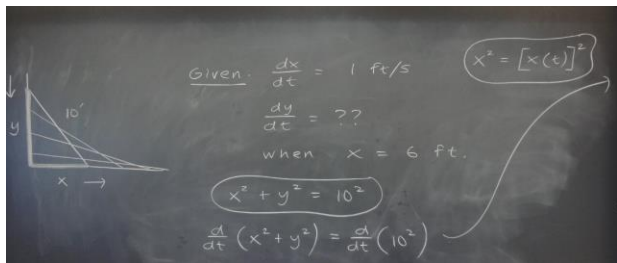
Using a number line:



2.8: Related Rates

WELCOME TO RELATED RATES

A LADDER 10 FT LONG RESTS AGAINST A VERTICAL WALL. IF THE BOTTOM OF THE LADDER SLIDES AWAY FROM THE WALL AT A RATE OF 1 FT/SEC, HOW FAST IS THE TOP OF THE LADDER SLIDING DOWN THE WALL WHEN THE BOTTOM OF THE LADDER IS 6 FEET FROM THE WALL?



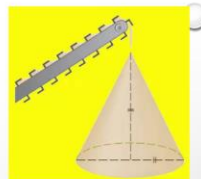
PROBLEM

As gravel is being poured into a conical pile, its volume V changes with time. As a result, the height h and radius r also change with time.

Knowing that at any moment $V = \frac{1}{3}\pi r^2 h$,

the relationship between the changes with respect to time in the volume, radius, and height is

- (a) $\frac{dV}{dt} = \frac{1}{3}\pi \left(2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right)$ (b) $\frac{dV}{dt} = \frac{1}{3}\pi \left(2r \frac{dr}{dt} \cdot \frac{dh}{dt} \right)$
 (c) $\frac{dV}{dt} = \frac{1}{3}\pi \left(2rh + r^2 \frac{dh}{dt} \right)$ (d) $\frac{dV}{dt} = \frac{1}{3}\pi \left(r^2(1) + 2r \frac{dr}{dh} h \right)$



Source: wnytutor

$$V = \frac{1}{3} \pi r^2 h$$

$(r(t))^2$
 $h(t)$

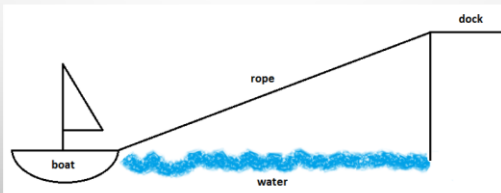
$$\frac{dV}{dt} = \frac{1}{3} \pi \left[r^2 \frac{dh}{dt} + h 2r' \frac{dr}{dt} \right]$$

From above, the answer is (a).

PROBLEM

A BOAT IS DRAWN CLOSE TO A DOCK BY PULLING IN A ROPE AS SHOWN. HOW IS THE RATE AT WHICH THE ROPE IS PULLED IN RELATED TO THE RATE AT WHICH THE BOAT APPROACHES THE DOCK?

- (a) ONE IS A CONSTANT MULTIPLE OF THE OTHER.
- (b) THEY ARE EQUAL.
- (c) IT DEPENDS ON HOW CLOSE THE BOAT IS TO THE DOCK.



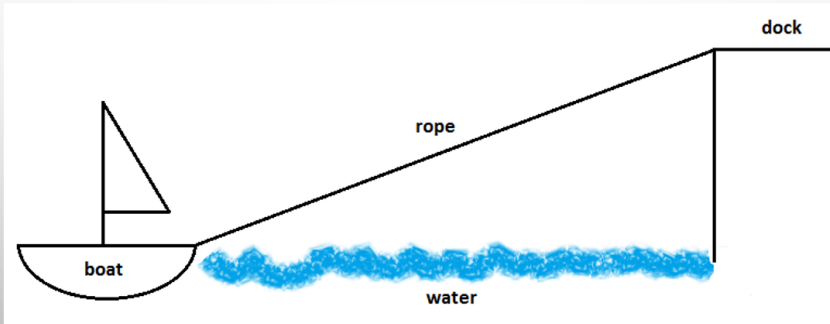
Source: Good Questions, Cornell University

The board to the right shows that the rate at which the boat is pulled in $\left(\frac{dy}{dt}\right)$ is related to the rate at which the boat approaches the dock $\left(\frac{dx}{dt}\right)$; the equation is $\frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt}$. The rates are not equal $\left(\frac{dy}{dt} \neq \frac{dx}{dt}\right)$ and they are not multiples of each other (for example, $\frac{dy}{dt} \neq 3 \frac{dx}{dt}$). The answer is (c).

Follow up question:

PROBLEM

A BOAT IS DRAWN CLOSE TO A DOCK BY PULLING IN THE ROPE AT A CONSTANT RATE. **TRUE OR FALSE:** THE CLOSER THE BOAT GETS TO THE DOCK, THE FASTER IT IS MOVING.



Source: Good Questions, Cornell University

pull the rope in
at a constant rate.
(GIVEN).

T/F: closer boat gets
to dock, the faster
it is moving.

$\frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt}$

$\frac{dx}{dt} \rightarrow \infty$

$x \rightarrow 0$

fixed

The diagram shows a boat on the left and a dock on the right. A rope is attached to the front of the boat and extends diagonally to the dock. The rope is labeled 'rope'. The boat is labeled 'boat'. The dock is labeled 'dock'. The water between them is labeled 'water' and is represented by a blue textured area.

(This problem picks up where the last one left off.) The statement is True (see work above).