Question 1

The literature in mathematics education makes explicit the benefits of tasks that promote advanced mathematical thinking (AMT). But after several decades of work, there is no widespread agreement on a precise definition of AMT, even after numerous attempts have been made to conceptualize (and reconceptualize) this construct. Based on these diverse perspectives, (a) What is advanced mathematical thinking? and (b) In what direction(s) might one turn to move the field forward?

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INTRODUCTION

Advanced mathematical thinking (AMT) is a thriving, contemporary research domain in mathematics education. From the mid 1980s to just prior to 2000, much work in AMT was published by the Working Group on Advanced Mathematical Thinking of PME (*International Group for the Psychology of Mathematics Education*). More recent work can be found in the collaborations by the Working Group on The Role of Advanced Mathematical Thinking in Mathematics Education Reform (PME—North American Chapter). Finally, from numerous publications in RUME (*Research in Undergraduate Mathematics Education*) to the vast collection of journal articles and book chapters, it is safe to say that the field has matured significantly from its mid 1980s state.[1]

Given this, a thorny issue remains at the heart of the work presented here. Despite the respect the field has earned, there is currently no widely accepted definition of AMT (Artigue, Batanero, & Kent, 2007; Selden & Selden, 2005). Quite simply, speaking to multiple authorities in the field about their perspectives may lead to conflicting characterizations. Below is just a sample of different views from some prominent workers in the field:

> If a student develops the ability to consciously make abstractions from mathematical situations, he has achieved an advanced level of mathematical thinking. (Dreyfus, 1991, p. 34)

> To observe and reflect upon the activities of advanced mathematical thinkers is in principle the only possible way to define advanced mathematical thinking. (Robert & Schwarzenberger, 1991, p. 127)

> The move to more advanced mathematical thinking involves a difficult transition, from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal deductions and their properties re-constructed through logical deductions. (Tall, 1992, p. 495)

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[1] One of the challenges in collecting research prior to 1985 is that much of the work did not appear under the auspices of "advanced mathematical thinking."

There seems to be another mode of thinking about mathematical concepts, a mode which has little to do with systematic deduction. This other mode is much more difficult to describe and to explain, but it is this special way of thinking which, according to many mathematicians, is the ultimate evidence of deep understanding. (Sfard, 1994, p. 49)

The parts of human cognition that generate advanced mathematics as an enterprise are normal adult cognitive capacities—for example, the capacity for conceptual metaphor. (Lakoff & Núñez, 2000, p. 351).

Mathematical thinking is advanced, if its development involves at least one of the . . . three conditions for an obstacle to be epistemological. (Harel & Sowder, 2005, p. 34)

Even withholding the context from which these quotes were drawn, one can detect surface distinctions in the researchers' perspectives. For example, Dreyfus speaks of the mental act of abstracting; Robert and Schwarzenberger perhaps about the work of professional mathematicians; Tall about the evolutionary shift from informal to formal; Lakoff and Núñez about the centrality of metaphor; Harel and Sowder about the cognitive obstacles that a learner must conquer. Upon closer examination, one may also find key similarities in the conceptualizations put forth. These positions as well as others will be explored in this paper.

The optimist would contend that the above variety is more a source of richness than anything else. However, such divergent views have generated much criticism, mostly encompassing the two issues below.

(a) What, specifically, is advanced? Is it the *thinking* or the *mathematics*? Perhaps both? (Artigue et al., 2007; Dreyfus, 1990; Harel & Sowder, 2005; Selden & Selden, 2005)

(b) The very nature of the word *advanced* generates inevitable comparisons to *elementary* mathematical thinking. Since the phrases themselves are value-laden, this is difficult to avoid.

Just a taste of the harsh criticism this has received can be found in Pimm (1995).

In conducting this literature review, the search for information began with five principal sources—David Tall's synthesis of five years of collaboration on the part of the PME working group resulting in the volume *Advanced Mathematical Thinking* (Tall, 1991a), Tommy Dreyfus's research synthesis of advanced mathematical thinking (Dreyfus, 1990), a research synthesis in the teaching and learning at the post-secondary level (Artigue et al., 2007), and the special issues of *Educational Studies in Mathematics* (1995) and *Mathematical Thinking and Learning* (2005)—each highlighting diverse views on AMT that are currently pervading the field. From these readings, many additional sources were retrieved and are included in this synthesis. In organizing and writing this review, it became evident that recent contributions are opening more doors than they are closing (Artigue et al., 2007). This is not a troublesome fact; it is the nature of the beast.

The purpose of this paper is the following. In reviewing the research on advanced mathematical thinking, the writer will first examine the different positions taken by the research community in describing—directly or indirectly—attributes, characteristics, or illustrations of AMT. In doing so, a rather detailed portrait unfolds. Naturally, praise and criticism will be offered where deemed appropriate and, of course, rationale for such feedback is provided. Next, an attempt to formulate a rudimentary definition of AMT based on these perspectives will be offered. Finally, a vision of a fitting direction for the next chapter in AMT research is given.

LITERATURE REVIEW

A brief note on the structure and organization of this review will prepare the reader for what is to come. First, a survey of the research is given under the title

Foundations—this is appropriate since this particular research either (a) serves as a support system for much of the work that followed, or (b) synthesizes elegantly what researchers had known for some time but had great difficulty articulating. Next, it seems fitting to discuss the variety of stage theories and other theoretical results that have emerged with respect to AMT; this section is called *Cognitive Theory*. Once this is done, the writer will survey the research that might be adequately placed into different schools of thought but relate closely to one or both of the aforementioned sections. These areas are called *Criteria for Advanced Mathematical Thinking*, *Linking Intuition with Formalism*, *Advanced Mathematical Practice: A Human Activity*, and *The Professional Mathematician*. The sequence of these four categories should not suggest a bias toward any category; this is simply what was found to be stylistically sound. Finally, brief summaries drawing on salient features follow each of the sections above. This is done in order to guide the reader through the developmental process of formulating a definition for advanced mathematical thinking.

<u>Foundations</u>

In an attempt to organize this section in a coherent way, it is convenient to discuss three areas that have been instrumental to the development of research in advanced mathematical thinking. These areas are *Concept Image and Concept Definition*, *Learning Obstacles*, and *Process-Concept Duality*.

Concept Image and Concept Definition

Although the work of Vinner & Hershkowitz (1980) first introduced the important constructs of *concept image* and *concept definition*, it stands that the particular

application of these ideas to limits and continuity (Tall & Vinner, 1981) remains the

staple from which much research has grown. First, the authors define *concept image* as

"the total cognitive structure that is associated with the concept, which includes all the

mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152).

For example, a student's concept image of *derivative* may include secant lines

approaching a tangent line, instantaneous rates of change, shortcut formulas, velocity, etc.

Clearly, these ideas are scattered and highly contextual—hence, falling short of defining

a cohesive whole. Nevertheless, given a new situation, learners may add or delete

something from their concept image.

On the other hand, a *concept definition*—as the name suggests—defines a concept

by either using informal ideas or formal mathematical language. Using the limit of a

function as a case in point, one may write

(a) $f(x)$ gets sufficiently close to L for x sufficiently close to c , or

(b) Given $\varepsilon > 0$, there is a $\delta > 0$ such that $\left| f(x) - L \right| < \varepsilon$ whenever

$0 < \left| x - c \right| < \delta$.

Statement (a) may be regarded as an informal definition whereas statement (b) aims to

pinpoint what "sufficiently close" might imply. Such definitions may be learned in a

passive state or through a meaningful experience. Tall and Vinner (1981) report

exclusively on how students' *concept images* for limits and continuity may contain

representations, pictures, expressions, and experiences that make it difficult to grapple

with the formal *concept definition*. Additionally, the authors dwell on issues of cognitive

conflict; these problems arise when image and definition fail to complement one another.[2]

Vinner and Dreyfus (1989) conducted a study of students' and teachers' conceptions of mathematical function. The participants included beginning college students across various disciplines as well as mathematics teachers at the junior high level. Participants completed a questionnaire whose aim was to reveal impression(s) of mathematical function including a personally meaningful definition, examples and nonexamples, and the degree of strength between image and definition. The final analysis included a total of 271 student responses and 36 teacher responses. Overall, it was found that participants rarely used definitions to classify functions (nor when attempting to construct functions). Instead, the individual's concept image was the determining factor in both cases. The authors make a reference to *compartmentalization* as defined in Vinner, Hershkowitz, and Bruckheimer (1981) in that the participants had two conflicting schemes of the notion of mathematical function.

In a similar way, Edwards and Ward (2004) make a distinction between *extracted definitions* (everyday usage) and *stipulated definitions* (mathematical) and use this framework to explain students' difficulty in using mathematical definitions, especially in the context of proof. Specifically, the authors were examining how students viewed the nature of and role played by definitions in real analysis and abstract algebra. Through analysis of written assignments, classroom observations, and task-based interviews, the researchers were trying to determine if undergraduates taking an abstract algebra course would encounter similar challenges in definition usage to those taking mathematical analysis. The results showed that, irrespective of the subject matter, students do not

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[2] This idea is discussed further in the next section *Learning Obstacles.*

categorize or use definitions like professional mathematicians. Some students see definitions as simply being "true" or dispose of the definition once a concept is "understood." Also, when concept definition and concept image paint different portraits, concept image seems to win over.

Vinner's 1991 study drives a crucial point home with striking clarity: Mathematics is a discipline built on its definitions and because of this, the subject is often taught in a manner which stresses the importance of knowing clear and precise definitions. However, teaching mathematics in this way ignores the cognitive and psychological aspects that are carried along with concept formation. In citing several research papers (Davis & Vinner, 1986; Tall & Vinner, 1981; Vinner, 1982, 1983), the author stresses that most students do not use concept definitions while engaged in tasks.[3] Instead, learners primarily rely on their thought habits connected to everyday experiences. An important message here is that precise definitions are of diminished importance once a person has a valid representation in his/her concept image. Vinner utilizes a scaffolding metaphor in that once a building is erect, there is no need for the scaffolding. The author makes a good point in that when a student learns something new, one of three things may happen:

    (a) The student may internalize this new information and add it to the existing concept image.

    (b) The student will only retain this for a short time and it will eventually get lost.

    (c) The student will deploy this new information when prompted.

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[3] Interestingly so, the author also discusses methodological details on how some questions in these studies were designed to elicit information about concept image while others were meant to provide information about concept definition.

Vinner calls situation (a) *accommodation* or *reconstruction*, borrowed from Piaget's genetic epistemology (Piaget, 1970). Clearly, this is the ideal situation. In scenario (b), the concept image is unchanged—the unfortunate reality. The third situation isn't much better since, when prompted for a concept definition, the learner will probably exclude this new information (Vinner, 1991).

This brief discussion on concept image and concept definition reveals that our concept image drives our actions. Vinner (1991) attempts to convey this point in arguing that "to understand, so we believe, means to have a concept image" (p. 69). As one may muse about the *contents* of such an image, Harel (1998) polishes this idea in writing "A student with an *effective* concept image is one who can communicate its corresponding concept definition in his or her own words, can think about it in general terms, . . ." (p. 499, emphasis added). These themes, as the reader will see shortly, play a critical role in teasing out the intricacies of advanced mathematical thinking.

Learning Obstacles

Consonant with the discussion of the conflict that can occur between concept image and concept definition, the work of Cornu (1991) does a decisively good job of portraying the many challenges of learning the notion of mathematical limit—from the inception and complexity of the idea, to the diversity of the individual's concept image, to the many cognitive obstacles that must be overcome. His early work (Cornu, 1983)[4] runs parallel to the notion of concept image; specifically, he introduced the idea of *spontaneous conceptions* as those views and perspectives that have been formed prior to

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[4] I regret that I have not read this work since it appears in French. However, many subsequent works were outgrowths of what appeared in his 1983 dissertation.

instruction. He posits that these ideas then blend with (but are not replaced by) formal

instruction. This amalgamation of informal knowledge and scientific theory can cause

contradictory notions to develop in one's concept image, so forming the seeds of future

learning obstacles. The cognitive obstacles discussed here are *genetic/psychological*

obstacles (pertaining to individual development), *didactical obstacles* (related to

instruction or the teacher), and *epistemological obstacles* (concerning the nature of the

subject and development of knowledge).

By quite a significant margin, the notion of *epistemological obstacle* has received

the most attention. It was first introduced by Gaston Bachelard in the 1930s, applied to

education by Brousseau (1997), and incorporated and refined by others (cf. Harel &

Sowder, 2005; Sierpińska, 1987). Bachelard believed that such obstacles were inevitable

barriers that stood in the way of knowledge acquisition. Even the pioneers who

developed fundamental mathematical ideas were not immune to epistemological

obstacles; this is evidenced by the obstacles' presence in the historical evolution and

quest for mathematical knowledge. As a particular illustration, Cornu (1991) recounts

the struggle of ancient geometers in their quest for the modern day conceptualization of

limit. In particular, the method of exhaustion relies on the specificity of the geometric

figures involved. Thus, the technique does not transfer to other settings. Since the

numerical relationships never crystallized, the limit concept was not developed at this

time. This "fundamental barrier" aligns with Brousseau's perspective that an

epistemological obstacle arises when an idea that works quite seamlessly in one setting

malfunctions in a different setting.

A key note here is that Cornu conveys the idea of "arbitrarily small" as an *object* resulting from encapsulation through the APOS theory[5] (cf. Dubinsky, 1991; Dubinsky & McDonald, 2001). Hence, when being introduced to the notion of a limit, the naïve learner must immediately come to terms with an idea that presumably requires the *a priori* mental experiences of *action* and *process*. Cornu believes this to be a central factor in explaining why limits are so difficult to grasp. He proposes a preliminary solution to this: introduce the limit idea informally so that the learner may internalize these experiences before formal instruction begins. This clearly resonates with the above discussion focusing on the nurturing of one's concept image. One way to do this that seems to be gaining momentum is through experiences with a computer (Artigue, 1991; Artigue et al., 2007; Dubinsky & Tall, 1991; Heid, 1988; Tall, 1992; Tall & Thomas, 1989). However, this new setting may present its own epistemological obstacles . . .

Process-Concept Duality

Gray and Tall (1994) allude, in many respects, to the nature of advanced mathematical thinking even if they avoid using the term. The authors hypothesize that highly successful individuals think, conceive, and reason about arithmetic in a way that differs markedly from their lower-achieving peers. They begin by discussing the inherent ambiguity in mathematics; that is, mathematicians often use symbols/notation such as '4 + 9' to mean several things: (a) the act of adding four and nine, (b) the process of addition, (c) counting four items followed by nine items and then determining how many there are, (d) the end result or product (i.e., the sum 13), etc. The successful thinker utilizes this ambiguity to flexibly think, reason, and manipulate through the task/*process*

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[5] The APOS (Action-Process-Object-Schema) theory is discussed in greater depth in the section *Cognitive Theory*.

resulting in the *concept* of "thirteen"—hence the blended term *procept*. Proceptual

thinking (i.e., a rich interwoven understanding of the process, product, procedures,

symbols, and concepts) might shed light on the inner workings of advanced mathematical

thinking. There are numerous references to the fact that professional mathematicians

seamlessly grasp this ambiguity of process and concept and then mentally compress these

notions for later usage:

> Instead of having to cope consciously with the duality of concept and process, the good mathematician thinks ambiguously about the symbolism for product and process. We contend that the mathematician simplifies matters by replacing the cognitive complexity of process-concept duality by the notational convenience of process-product ambiguity. (Gray & Tall, 1994, p. 121).

The authors provide empirical evidence that proceptual thinkers actually lessen their

cognitive load when solving complex problems since they have a rich core of mental

images whereas the procedural thinker might resort to primitive techniques of counting.

This evidence appears in (a) the difference in *strategy* used by above-average and below-

average children and (b) the detailed accounts of the researcher's one-on-one interviews

with several children. The distinctions, which often mark the difference between success

and failure, is coined the *proceptual divide*.

It seems only fair to mention that the marriage of the two ideas 'process' and

'concept'[6] had been explored in earlier works (Dubinsky & Harel, 1992; Harel & Kaput,

1991; Schwarzenberger & Tall, 1978; Sfard, 1991) but the impression that this particular

research (Gray & Tall, 1994) left on the math community is unmistakable: This

ambiguity and flexibility might be fostered from an early age, not just in tertiary-level

"advanced" mathematics per se; see Harel and Sowder (2005) for more on this.

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[6] Sometimes the word 'object' is used instead of 'concept' yet they appear to have similar meanings.

Summary

The ideas presented in this opening section are not to be interpreted as synonymous with advanced mathematical thinking. They do, however, provide the pillars for what it means to unpack mathematical knowledge and to build mathematical understanding. The unfortunate divorce of concept image and concept definition in the learner's mind may lead to compartmentalization, only exacerbating the effects of the didactical and psychological obstacles that lay ahead. However, proceptual thinking may afford the learner opportunities to deal with these obstacles (even epistemological obstacles) in efficient ways, thus opening the doors to advanced mathematical thinking. Perhaps the greatest contribution of this research is its explicit reference of mathematical learning as personal and idiosyncratic. In the definition of AMT proposed at the end of this report, this influence is conveyed by framing AMT as a phenomenon *in the mind* of the learner. Furthermore, it is contrasted with additional research which will be discussed at a later point in this paper.

Cognitive Theory

Although there is no shortage of theoretical developments in reference to concept formation in AMT, the theories of Dubinsky and colleagues (Dubinsky, 1991; Dubinsky & McDonald, 2001), Sfard (1991), and the more recent reconceptualizations from Sfard (2001, 2006, 2008) have received the most attention. Subsequent theories (cf. Chin, 2003; Chin & Tall, 2000; Dubinsky & Harel, 1992; Gray, Pinto, Pitta & Tall, 1999) appear to use the theories from the 90s as springboards for further developments. Because of this, the writer will provide detailed descriptions of the theories of Dubinsky and Sfard and then proceed to explain the other theories as merely applications of these.

APOS Theory

The APOS (Action-Process-Object-Schema) theory put forth by Dubinsky and colleagues (Dubinsky, 1991; Dubinsky & McDonald, 2001) is generally conceived as both an ambitious and successful contribution to our understanding of concept formation in advanced mathematical thinking (Pimm, 1995; Tall, 1999; Thompson, 1993). Dubinsky (1991) attempts to extend Piaget's theory of cognitive development known as *reflective abstraction* to advanced mathematical thinking (Piaget, 1976, 1978). Initially, Piaget's theory had only been deemed appropriate to topics such as proportion and arithmetic but Dubinsky extends this theory to topics typically encountered at the tertiary level.

Piaget's cognitive theory makes a distinction between *empirical abstraction*, *pseudo-empirical abstraction*, and *reflective abstraction*. In empirical abstraction, the learner focuses on concrete objects and strives to obtain knowledge about them. Pseudo-empirical abstraction bridges this with more abstract reasoning as the learner may act on these external objects and/or imagine such actions. Finally, reflective abstraction is the culmination in which the objects manipulated and/or created exist only in the learner's mind.[7] The outcome of this stage is that the learner develops new actions and new objects—from this, the cycle of empirical/pseudo-empirical/reflective abstraction

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[7] An example of pseudo-empirical abstraction might be seen in the popular M&M experiment for exponential functions. Here, an individual shakes four M&M's in a cup, empties out the candies, observes the number of M's face up, adds precisely this many candies to the cup, shakes, empties, and repeats. Recording data in this manner, a student may begin to see characteristics of a good approximation of the function $y = 4 \cdot (1.5)^x$. Although the learning of the exponential function $y = 4 \cdot (1.5)^x$ may be associated with the M&M's themselves (thus inherently empirical), this is an intermediate step in grasping the generality of the exponential function $y = a \cdot b^x$ as a mental object that exists in the mind, subject to manipulation (reflective abstraction).

resumes. This feedback system continuously replenishes itself into the development of a schema (Dubinsky, 1991; Piaget, 1976, 1978).

Dubinsky discusses five kinds of knowledge construction (interiorization, coordination, encapsulation, generalization, and reversal) within Piaget's notion of reflective abstraction; these ideas tie together his APOS theory.[8] In an attempt to be brief, *interiorization* involves internalizing and translating phenomena into cognitive entities, *coordination* is the molding and blending of several processes into a new process, *encapsulation* converts process into object, *generalization* is the application of a schema to a broader context, and *reversal* develops new processes by undoing existing ones. An illustration of a specific mathematical idea (e.g., function) as seen through the lens of this theory may assist the reader:

Action: A function is a formula. Replace *x* with 5 to obtain a value.

Process: A function is an input/output machine. Repeat and reflect on the action and this leads to process. This is akin to what Harel calls *repeated reasoning* in his DNR theory (Harel, 1998, 2007).

Object: One can operate on and/or blend functions to obtain new functions (e.g., transformation, composition, etc.). The totality of the function concept comes into focus.

Schema: This is an organized and coherent assortment of the above mental constructions that can be triggered in problem-solving situations involving functions.

In sum, the learner interiorizes an action; this becomes a process. Through a series of coordinations and possibly reversals, these processes feed back into the construction of new objects by way of encapsulation. This process continues, thus building into the

[8] *Reversal* was used by Piaget but not included in his notion of reflective abstraction.

development of a richer schema (i.e., organized concept image) by way of the learner generalizing. That is, he/she applies and tests the schema in novel situations and either adjusts it accordingly and/or lets new information shape the schema.

Operational and Structural Conceptions

In Sfard (1991), a beautifully elaborate stage theory on the concept formation of mathematical ideas is presented. The overarching claim is that *operational* conception and *structural* conception—though seemingly incompatible ways of understanding mathematical ideas—together form the basis for *reification*. Reification occurs when *processes* become *objects*; these objects are then the future players in interiorization.[9] In this paper, an *operational* conception is dynamic in nature—one in which the learner views the idea as a process resulting from actions (e.g., applying an algorithm). In contrast, a *structural* conception is held when the learner views the idea in a static fashion—as an object or entity that can be manipulated. It is Sfard's belief that, in general, operational conceptions precede structural conceptions and she provides a wealth of evidence from a historical perspective to support her position.

Sfard's theory occurs in stages in which the learner first views ideas as *operational*. Following this, the learner progresses through the stages of *interiorization*, *condensation*, and *reification*—thus achieving a *structural* conception. A brief description of each term follows. On the operational side of the coin (to borrow Sfard's metaphor), *interiorization* amounts to the learner familiarizing himself with processes; increased proficiency is usually the result. As the learner starts to see processes as more general, *condensation* allows the learner to become less concerned with details as the

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[9] *Interiorization* is used here in a similar way as seen in the APOS theory. However, the word was first popularized by Piaget (1970) in the context of genetic epistemology.

whole entity comes into focus (even if the mathematical idea is still tightly linked to the process conception). Finally, the structural side of the coin culminates in *reification,* best described by Sfard herself:

> Only when a person becomes capable of conceiving the notion as a fully-fledged object, we shall say that the concept has been reified. *Reification*, therefore, is defined as an ontological shift—a sudden ability to see something familiar in a totally new light. (Sfard, 1991, p. 19).

In other words, what was initially a *process* spontaneously congeals into a static *object*. With this newly formulated structural conception, objects now become the tools for further interiorization.

A natural progression of this work is found in Sfard (1994) where she asserts that reification is the "birth of metaphor" en route to developing deep understandings of mathematical ideas. The central hypothesis of this paper is that thinking in mathematics occurs primarily through metaphors embodied in experience (cf. Johnson, 1987; Lakoff & Johnson, 1980; Lakoff & Nunez, 2000). This theory builds on the assumption that the development of abstract mathematical ideas (even those that seem to have no association with real-life phenomena) come into being by way of association with past experiences. Moreover, these mentally constructed metaphors shape new directions, give meaning to abstract ideas, and allow for imagination and reasoning in mathematics. Consequently, these abstractions inherit structural properties of worldly entities even while possessing no direct association to the concrete. On the one hand, it is what allows thinkers to give meaning to what would otherwise remain intangible and abstract. But on the other hand, it confines our capacity and ability to precisely these perceptual embodiments. This, in turn, explains the challenge(s) of reification.

For the reader's convenience, the theories of Dubinsky and Sfard are now displayed diagrammatically as the authors did in the original works. In the case of Sfard, the language of Hadamard[10] (1949) supplements the model. See the following two pages.
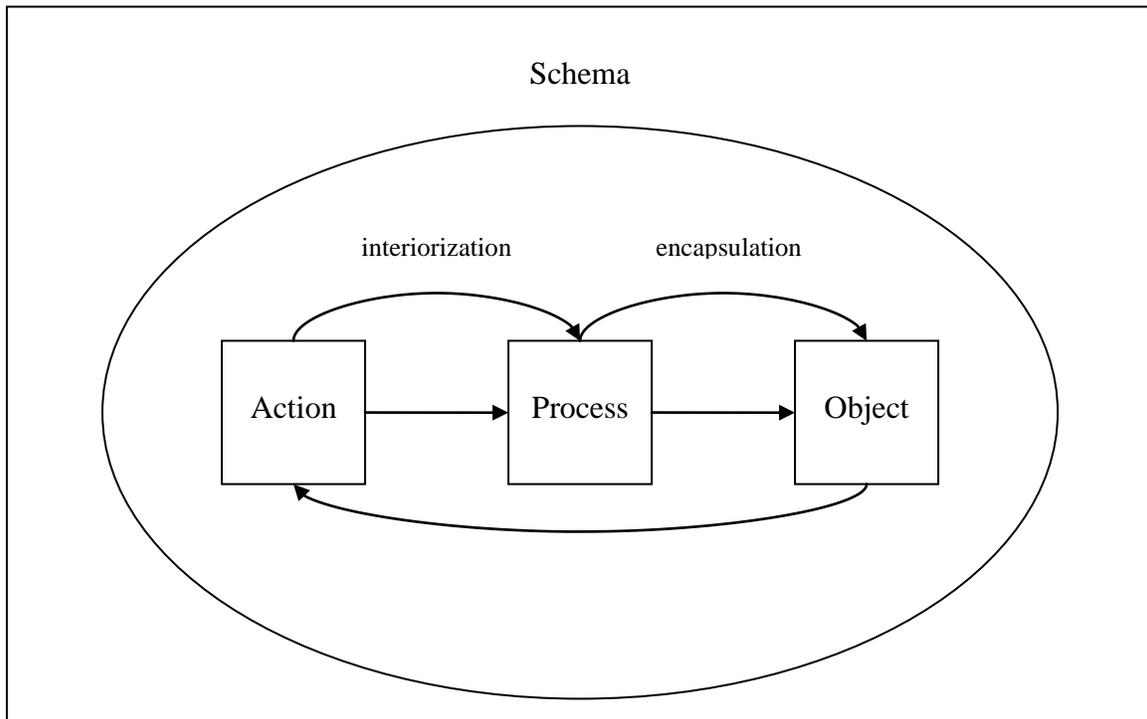

Figure 1. Dubinsky's APOS Theory

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[10] Hadamard (1949) once used the words *incubation* and *illumination* to describe what Sfard later called *interiorization/condensation* and *reification*, respectively (Sfard, 1991).
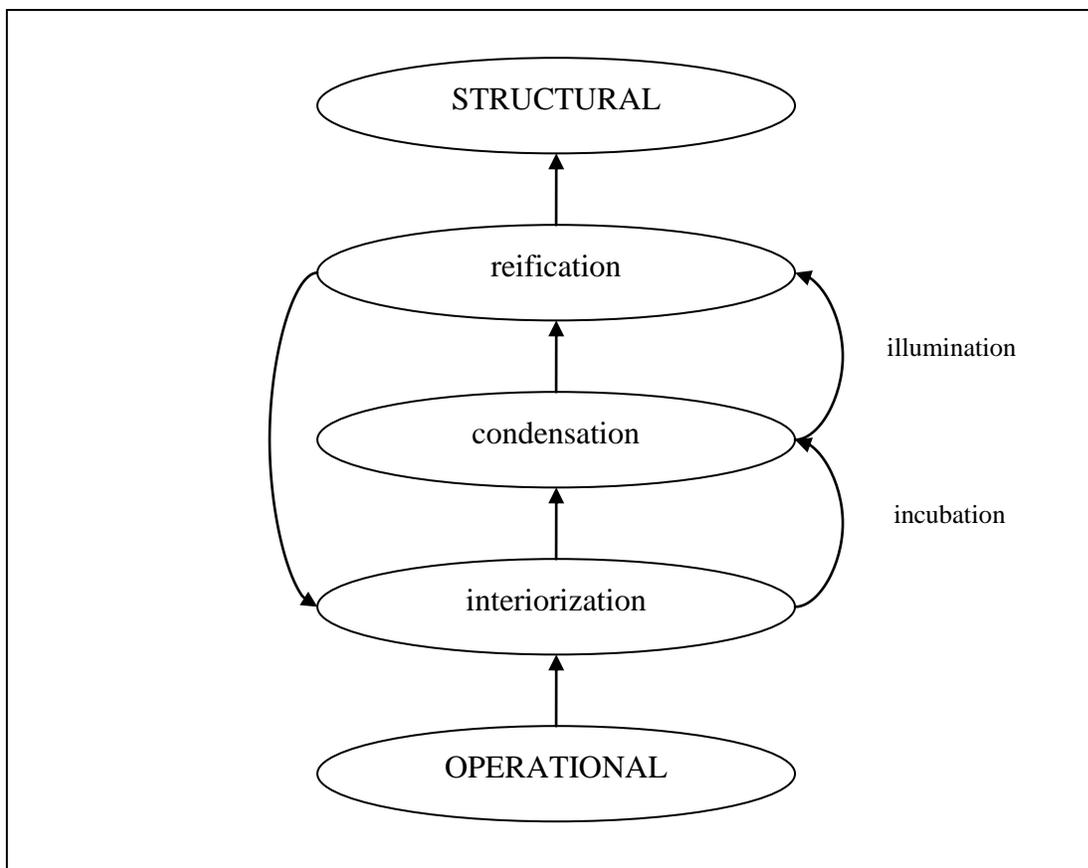
Figure 2. Sfard's Operational/Structural Theory

It would be negligent to omit other theories that have emerged as research in advanced mathematical thinking has matured. For example, Chin (2003) and Chin and Tall (2000) discuss a stage theory outlining a hierarchy of reasoning and thinking in the context of mathematical proof; Ervynck (1991) presents an intricate stage theory leading to mathematical creativity; Dubinsky and Harel (1992) discuss students' conception of function in terms of a progression from *prefunction* to *action* to *process* to *object*; Gray, Pinto, Pitta, and Tall (1999) develop a theory based on the principles of *perception*, *action* and *reflection*—one that embraces the fact that depending on the nature of the subject at hand and the individual's personal preferences, learners construct their own knowledge with varying degrees of success. The rationale for not discussing these

theories further is rooted in their specificity to particular areas of mathematics. This

limits their applicability to some degree. With the exception of Tall's (1999) critical

reflection on the limitations of the APOS theory and Confrey and Costa's (1996) critique

of the failures to address social factors in learning, the two theories presented here have

been well received and are universal in their applicability in advanced mathematical

thinking research.

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Participationist Discourse: Thinking as Communicating[11]

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Anna Sfard has been leading a recent movement to reconceptualize the nature of

thinking in mathematical contexts (Sfard, 2001, 2006, 2008). Grounded in sociocultural

traditions and the communal nature of human learning, *thinking* is described as a form of

communication through discourses she calls "participationist".[12] To begin, thinking is

still considered as deeply interpersonal but it differs from traditional cognitive science by

way of its equivalence to *private communication* with the self. In contrast to an

acquisitionist view that asserts "good" mathematical thinking from resolving cognitive

conflict or finding coherence with difficult situations (cf. Brousseau, 1997; Edwards &

Ward, 2004; Harel & Sowder, 2005; Tall & Vinner, 1981), Sfard embraces

communication as the cornerstone to mathematical thinking. This results in coordination,

mutual growth, and dexterity between and across participants—leading to an

accumulation of (mathematical) knowledge for the community.

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[11] Some might argue that Sfard's recent attempt(s) to describe mathematical thinking as "communication" (Sfard, 2001, 2006, 2008) fall outside the boundaries of traditional work in the AMT strand. However, it is precisely for this reason—the radical nature of casting *thinking* as *social discourse*—that this view is especially interesting. Moreover, she is the first to admit that her reconceptualization of thinking is likely to "raise some eyebrows" (Sfard, 2001, p. 50) given its departure from traditional cognitive science including some of her own contributions (cf. Sfard, 1991).

[12] To be clear, Sfard (2001) defines *discourse* as any example of communication—internal or external, verbal or symbolic, with other individuals or alone, etc.

To stress the fuzzy boundary between cognition and communication, she blends the two terms in her study of *commognition*—a "term that encompasses thinking (individual cognition) and (interpersonal) communicating; as a combination of the words communication and cognition, it stresses the fact that these two processes are different (intrapersonal and interpersonal) manifestations of the same phenomenon." (Sfard, 2008, p. 296). It is here where the acquisitionist paradigm—prominently embraced in most cognitive studies from the 80s and 90s—falls short in explaining *why* students have specific shortcomings in their understanding. An example from Sfard's work is explained presently for illustration.

Sfard begins by asking the most ubiquitous question in mathematics education: *Why do some students successfully learn mathematics while others do not?* To address this, she provides a glimpse of a transcript taken from a previous study in which two students are exchanging thoughts about tasks related to the idea of mathematical function (Sfard, 2001, 2008). From the dialogue between the students, it is clear that one student, Ari, is making great strides in completing the task while another student, Gur, struggles to make connections between different representations (tabular and symbolic). Viewed from an acquisitionist perspective, Sfard tells us that Gur may have "underdeveloped schemes" or that concept definition and concept image may be in full clash. Regardless, the bottom line is that the mathematical function *idea* eludes Gur. Despite this position, Sfard reminds us that having such information lacks the explanatory power to address what actions teachers might take to help Gur address such problems. In fact, her theory emphasizes that the "problem" may not be with Gur at all. Instead, Sfard identifies the

nature of the interaction between the participants as the primary unit of analysis. This effectively influences Gur's self-communication and thus, his commognition.

Viewed through the lens of *thinking as communicating*, Sfard revisits this episode and identifies a dysfunctional discourse as the root problem. She asserts that Gur's "lack of understanding" is cemented in the pair's inability to communicate with one another effectively. In short, all communication efforts are foiled by several factors: Gur's lack of focus, his hopeless quest in search of surface clues (that will lead to his mathematical growth), his need to save face, and his intense focus on interpreting Ari's speech to the chagrin of ignoring his own thinking. In the end, the collective unit of "Ari and Gur" paints the *antithesis* of skillful participants in communication since (a) Ari's discourse is objectified[13] while Gur's is not, (b) the two participants clash with respect to discursive focus (i.e., in their exchanges, the students often attend to different foci even if their language is identical), and (c) Ari is somewhat unforgiving and indifferent to the needs of Gur. As Sfard asserts, "it takes the two to produce a failure." (Sfard, 2001, p. 44). Although this position is quite provocative and likely to spur some heated discussion, it introduces an attractive focal point (i.e., communication) to address reform in mathematics education—one that is far removed from the view of "fixing" students' naïve/faulty conceptions.[14] Many other examples can be found in Sfard (2001) and Sfard (2008).

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[13] The term *objectified* is intended to mean that Ari's conception is one of the structural nature (Sfard, 1991).

[14] However, what remains unsatisfying is something most mathematics educators would tacitly infer from the written transcript between the two students. That is, if these students attempted the written exercise *individually*, Ari would presumably "succeed" and Gur would presumably "fail." That is, regardless of communication between the two, the outcome of Sfard's initial question appears unchanged. From this perspective, it appears that the notion of "what is in the mind" more sharply illuminates Gur's difficulties.

In sum, Sfard asserts the importance of collective outward communication which pushes participants to a higher plane of knowledge spurring further communicative acts. Different branches of communication create factions of discourses but it is by way of *individualizing* these discourses (through making it one's own) that leads to mathematical learning. She makes a strong case for how "outsiders" who are not yet fluent in these circles may initially see mathematical thinking/activity as ritualistic (Sfard, 2006). By engaging in communities of discourse where amateur peers interact with experts (and these experts may be peers as well), the learner becomes proficient in these discourses en route to individualization. Embracing this point of view, it seems that "advanced mathematical thinking" might align with what Sfard would today call an *objectified discourse*. Since her work is ongoing, only time can determine its reception in the eyes of mathematics educators across the globe.

<div align="center">*Summary*</div>

There are several points to explicate in reviewing these theories. First, the similarities between Dubinsky's APOS theory and Sfard's operational/structural theory cannot be overlooked. Beside the deep architectural connections, the particular stages even resonate with similarities. For example, Dubinsky's *process* and Sfard's *condensation* share parallel qualities; likewise, *reifying* is analogous to *encapsulating*; finally, Dubinsky's cycle of abstraction mirrors the flow of Sfard's duality seen in lower-level reification and higher-level interiorization.

Second, the notion of schema adjoins a notable quality to concept image as discussed previously. As Dubinsky (1991) posits, "A schema is a more or less coherent collection of objects and processes." (p. 102). From this, one may perceive a close

relationship between reflective abstraction and the contents of one's concept image.

Specifically, it appears that reflective abstraction hints at one's ability to organize,

retrieve and apply information from the concept image in a meaningful way (since the

information in one's concept image need not be coherent). Furthermore, this mode of

thinking results in a considerable change: When a new situation is encountered, the

organized concept image (i.e., one's schema) adjusts accordingly.

A third point of both theories is the apparent foreshadowing of elementary

procepts as discussed in Gray and Tall (1994). Sfard provides several examples in which

viewing a mathematical notion through both lenses is immensely beneficial to the learner.

As a case in point, a function can be viewed as a set of ordered pairs (object) or as an

input/output machine (process). She also articulates how mathematical representations

have the potential to forge connections between these two modes. That is, standard

functional notation suggests a computation by way of a formula (operational) but also

alludes to a static relationship as a statement of unchanging identity (structural).

Similarly, Dubinsky mentions the advantage of viewing some mathematical ideas as both

process and object—again, a clear precursor to the flexibility of proceptual thinking.

Finally, the theories are surprisingly robust in their capacity to elucidate the lulls

in mathematical development. In discussing the historical evolution of mathematics and

the struggles therein, one cannot help but be drawn to the work of Brousseau and Cornu.

One can see the significance of why epistemological obstacles are deeply rooted in the

historical development of the subject (Brousseau, 1997). After all, it seems sensible that

students encountering mathematics for the first time would face similar challenges to

those who wrestled with its contents centuries ago. For example, Cornu's discussion of

the limit idea never crystallizing may be perceived as a failed effort to reify a mathematical process as a structural entity (Cornu, 1991). In sum, the theories stretch beyond their original range of extrapolation in offering credible explanations for the struggles in contemporary mathematics.

Where both the APOS and operational/structural theories fall short is in their omission of social factors in shaping mathematical thinking. Most AMT research has, at the heart, the learner as understanding concepts, refining them, and building rich schemas from these experiences. In turn, learners accumulate pieces of knowledge and connect them to enrich their understanding of the world. We acquire such knowledge and thus "own" it as if it were a commodity (Sfard, 1998). In contrast, a social view would place *apprenticeship* at its center so that the learner is viewed as one of many in a community of discourse. "Becoming a participant" is consonant to "acquiring understanding" while "building community" is analogous to "personal enrichment" (Sfard, 1998). From this perspective, it is clear how Sfard's newer framework of "thinking as communicating" has filled a void.

With respect to the developmental process of formulating a definition of AMT, two factors from this section rise to the surface. First, the emphases on objects and processes suggest multiple states in mathematical thinking akin to the *static* and *dynamic* features in mathematical content. Second, Sfard's newer developments challenge us to consider communication as a form of thinking—suggesting that the activities in which we engage and the communication across participants (and the self) together define the essence of our thinking.

<u>Criteria for Advanced Mathematical Thinking</u>

In this section, the writer will discuss a class of research papers that attempt to characterize advanced mathematical thinking as possessing certain properties. Some authors question the existence and/or contents of such a "list of characteristics" while others are bold enough to extend their viewpoint. Still others circumvent the issue by explaining what AMT is *not* while some choose to emphasize what makes it unique.

Edwards, Dubinsky, and McDonald (2005) define AMT as thinking that requires deductive and rigorous reasoning about concepts that are inaccessible through our five senses. The authors are keen in stressing the importance of *all three* properties (i.e., deduction, rigor, and inaccessibility); they offer examples of thinking that evoke only two but fall short of AMT. Their definition is broad enough so that AMT can occur in a variety of educational experiences (perhaps nonmathematical) and at various levels of mathematics. Specifically, the third component of their definition can be interpreted as a form of reflective abstraction, so bringing to the fore Dubinsky's APOS theory. Likewise, Sfard (1991) mentions that humans' inability to access abstract ideas through the senses—so one's ability to *envision* such objects—may play an important role in the mathematical experience.

Harel and Sowder (2005) point to Brousseau's work on epistemological obstacles as forming the backbone to a suitable definition for AMT (Brousseau, 1997). As previously mentioned, an epistemological obstacle is a piece of knowledge that stands in the way of acquiring new knowledge as it hinders the progress of those who attempt to overgeneralize. These obstacles (a) prove valid only in certain situations, (b) are resistant to the occasional contradiction, and (c) are deeply rooted in the historical fabric of

mathematics development (Harel & Sowder, 2005). The authors believe that thinking is "advanced" if its progression involves at least one of the notions above.

Harel and Tall (1991) put forth three principles of generic abstraction—the entification principle, the necessity principle, and the parallel principle. The *entification principle* states that learning models should contain objects that students can manipulate so that meaningful abstraction can occur. The *necessity principle* states that students must see a rationale for an idea. Finally, the *parallel principle* allows one to abstract from a specific situation more complex generalizations without adhering to the particulars of the specific situation. This abstraction is done via *expansive generalization*—that is, the learner extends his/her existing schema to broader situations (without reconstructing it). The authors further describe *reconstructive generalization* as calling for mental reconstruction prior to expanding one's range of application and *disjunctive generalization* as formulating new disjoint schemas to cover a context that is (seemingly) new to the learner. The researchers use systems of equations in one, two, three, and *m* variables to support their arguments.

It can be inferred that the generalizations discussed here—expansive, reconstructive, and disjunctive—progress from most favorable to least favorable and cognitively easier to cognitively more difficult, respectively, for the learner. The authors posit that reconstruction is sometimes necessary to achieve expansive generalization. This can be seen in the work of Biza and Zachariades (2006) where it is argued that misconceptions are synthetic models created by the learner for coping with situations which may be inconsistent with concepts as derived from his/her concept image.[15] By means of the theory of Harel and Tall (1991), the authors assert that AMT occurs in

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[15] This work is focused specifically on students' conceptions of tangent line.

conjunction with reconstructive generalization—that is, when the "concept image has to be radically changed so as to be applicable in a broader context." (p. 169). It is believed that the preference of reconstruction on the part of Biza and Zachariades is due to students having already internalized an overly-simplified representation of tangent line—most likely the effects of a didactical obstacle.

Other researchers approach the task of characterizing AMT by illustrating cases which are the *antithesis* of this type of thinking. For example, in a study by Selden, Mason, and Selden (1989), average calculus students (earning a C grade in first quarter calculus) were given problems that the authors defined as *cognitively nontrivial*. That is, the students had no prescription to solve the problems, nor were they taught any specific method beforehand. In general, the problems asked students to apply very familiar ideas from calculus but in a novel way. For example, one problem called for a solution to the equation $4x^3 - x^4 = 30$ or to explain why no solution could be found. Although many might view this as an algebra problem, a quick application of optimization theory leads to an answer. The results of the study were sobering in that the students had virtually no success and relied on previously learned (elementary) mathematics in their attempts to find solutions.

From a different perspective, Gray et al. (1999) explain AMT in terms of what sets it apart from elementary mathematical thinking (EMT). They posit that a key factor separating the two is that EMT begins with the study of objects and then derives its properties (e.g., geometry) whereas AMT begins with properties (e.g., axioms) with the aim of creating objects (groups, vector spaces). "This didactic reversal—constructing a

mental object from 'known' properties, instead of constructing properties from 'known' objects causes new kinds of cognitive difficulty." (Gray et al., 1999, p. 117).

Finally, Lithner (2003) studied students' ways of thinking as they attempted to solve textbook exercises in calculus. Borrowing heavily from the work of Polya (1990), the author defines *plausible reasoning* as a way of thinking that is grounded in the examination of "intrinsic mathematical properties" (Lithner, 2003, p. 33). In contrast, reasoning derived on *established experience* is rooted in the learner's past encounters with mathematics. Finally, students may reason by *identifying similarities*; that is, the individual will haphazardly attempt to identify features in a problem similar to one previously encountered, often resulting in artificial understanding (Lithner, 2003).

In this study, three students (separately) worked through several calculus exercises while the author videotaped the session; an interview with the individuals was held afterward. It was found that *identifying similarities* was a dominant course of action for many of the exercises. Even when inappropriate or unnecessary, the students sought examples in which surface similarities could assist in solving problems. The need to mimic previously learned procedures strongly suggests an inability or perhaps unwillingness to take part in constructive reasoning. Lithner is careful in mentioning that these habits—rather than originating in the students' minds—may have their roots in mathematics instruction. Although the author does not explicitly state this claim, one may infer a hierarchy in the sense that *plausible reasoning* sits atop the heap with *established experience* and *identifying similarities* playing subordinate roles. All seem to fall short of characterizing AMT but they appear to suggest a chain toward defining it.

Summary

In the struggle to define advanced mathematical thinking, it is inevitable that someone would eventually ask, "What constitutes advanced mathematical thinking? Is advanced mathematical thinking in some sense 'generic' or does it have particular characteristics within particular content areas in mathematics?" (Heid, Ferrini-Mundy, Graham & Harel, 1998, p. 54). When synthesizing the viewpoints put forth in this section, it is clear that the first question still has no definitive answer. Advanced mathematical thinking unmistakably means different things to different people. It might be characterized as generic in some writings (Edwards et al., 2005; Harel & Sowder, 2005; Harel & Tall, 1991) but tailored to the more specific in others (Biza & Zachariades, 2006; Chin, 2003; Chin & Tall, 2000).

In light of the research volume *Advanced Mathematical Thinking* (Tall, 1991a), one finds the final third of the book focused on the challenges in specific areas of mathematics, thus adding more credence to the view that AMT depends on the particulars of the content (Alibert & Thomas, 1991; Artigue, 1991; Cornu, 1991; Eisenberg, 1991; Tirosh, 1991). However, in recent years, increased attention has been given to bridging the divide between K-12 and undergraduate education,[16] thus supporting the view that AMT be promoted from a young age (Dreyfus, 1990; Gray & Tall, 1994; Harel & Tall, 2005; Maher & Martino, 1996; Rasmussen, Zandieh, King & Teppo, 2005; Selden & Selden, 2005). This latter view clearly necessitates a broader scope.

The perspectives summarized in this section add much richness to the debate encircling AMT, but at the same time, it illustrates why this type of thinking is so difficult to capture in a form that equally accommodates all voices. In fact, the recent

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[16] This was one of the driving forces in the formation of RUME.

assertion of AMT as a general construct in need of attention across the K-16 grades may

be too "all-purpose."   A consequence of this is a definition of AMT that merely provides

a heuristic account of "abstracting" and "generalizing"—missing the importance of the

mathematics on hand. This concern is addressed in discussing the definition of AMT

found at the end of this report.

<div align="center">Linking Intuition with Formalism</div>

The words from David Tall's 1992 handbook chapter (repeated from the

introduction) capture the essence of what this section is about:

> The move to more advanced mathematical thinking involves a difficult transition, from a
> position where concepts have an intuitive basis founded on experience, to one where they are
> specified by formal definitions and their properties re-constructed through logical deductions.
> (Tall, 1992, p. 495).

For decades, educational researchers have dwelled on the importance of linking informal

knowledge with mathematical formalism. Such associations are evident from the earlier

discussion on concept image and concept definition (Tall & Vinner, 1981), spontaneous

conception (Cornu, 1991), and extracted and stipulated definitions (Edwards & Ward,

2004). In this section, additional research with a similar scope is discussed as a means of

communicating this bond.

The work of Alibert and Thomas (1991) advocates the use of Leron's (1983,

1985) notion of *structural proof* in connecting formal and informal modes of thought.

That is, informal blueprints of proof *structure* may help provide a pathway to guiding

one's thinking.[17] Although more time-consuming and less elegant than the proofs one

would find in conventional math textbooks, the net gain of structural proofs is

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[17] Additionally, the intuitive construction of mathematical objects may prove instrumental to writing and
comprehending mathematical arguments.

unmistakably clear: Such proofs divulge the very ideas that are often masked in the layout of 'finished' proofs. The recent work of Weber (2004) gives a striking account of this.

Consistent with Vinner's (1991) finding of students' overdependence on the habits of daily experience, Tirosh (1991) found that students' intuitions about infinity are largely erroneous generalizations arising from manipulating finite entities in a finite world. To combat these perceptions, the author developed an instructional unit on Cantorian set theory which embraced the following components: (a) explicit treatment of conflicts and inconsistencies in student thinking/reasoning, (b) discussions of intuitions about infinity, (c) treatment of the similarities and differences between the finite and infinite, and (d) historical accounts of the negotiations that were reached in mathematical developments. Two questionnaires were given on two separate occasions over the course of the instructional unit. One questionnaire was designed to determine whether the students were able to apply what they had learned while the second provided information on how intuitions were influenced by instruction. The impact of this treatment was two-fold since most students (a) arrived at the self-realization of the inadequacy of intuition alone, and (b) saw the need for formal proof with respect to set theory.

It is interesting that, time and again, studies reveal humans' natural tendency to tap into previous knowledge bases when faced with unfamiliar situations. In the work of Edwards and Ward (2004), one of the authors felt that student difficulties in definition usage in *analysis* would contrast with that seen in *abstract algebra*; this hypothesis was grounded in the fact that words like "continuity" and "limit" might be confused with everyday usage but that terms such as "group" and "coset" would not. This conjecture

was refuted when it was found that students accessed incorrect knowledge in attempting to build a concept image of an idea that was initially unfamiliar (e.g., coset). There is a growing body of research indicating that intuition in mathematics—at least for some individuals—can stand in the way of knowledge acquisition (Edwards & Ward, 2004; Tirosh, 1991; Vinner, 1991).

The above studies sit in stark contrast to those which see intuition as central to mathematical activity (cf. Lakoff & Núñez, 2000; Núñez, Edwards, & Matos, 1999; Pinto & Tall, 2002; Tall, 2001). Here, informal thinking and intuition provide grounding for abstract ideas. Pinto and Tall (2002) provide some interesting reflections on the idea of *thought experiment*—a 'natural' way of human thinking free of formalisms and technical jargon. They offer their interpretation of Euclidean geometry as a product of thought experiments.[18] Here, concrete examples serve as prototypes for more general situations and the authors posit that learners perceive general notions as embodied within particular examples. As a case in point, the general law of "multiplication commutes" is evidenced through examination of a 2 by 3 array and a 3 by 2 array both having 6 objects.[19] Again, the emphasis is on how informal thought experiments and visual imagery can serve as a platform for giving meaning to formal mathematics. Tall (2001) even proposes a reversal in direction by theorizing that mental imagery may clear a pathway for formal construction and axiomatic theory.

Lakoff and Núñez (2000) provide a refreshingly unique perspective on the origin of mathematics and how humans come to understand it. They argue that the bulk of mathematics is understood and filtered through our unconscious system of thought. That

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[18] For example, many of the theorems can be plainly 'seen' as true by looking at a picture or a specific situation. Then by adopting a shared language, Euclidean geometry is born.

[19] The similarity to the *parallel principle* conveyed by Harel and Tall (1991) is noted.

is, just as humans use analogies and imagery to convey meaning in everyday life, abstract mathematical ideas are concretized by conceptual metaphorical mappings; this they call *mathematical idea analysis*. For example, they cite '*size = importance*' as one such metaphor. "This is a big day for me" might indicate much at stake whereas "It's a small matter" hints at a trivial situation. Such expressions are so deeply engrained in our biology that we rarely stop to think of their literal meaning.[20] The authors provide a wealth of additional examples backed by two decades of research in cognitive science and linguistics.

As an illustration, the abstract notion of "set belongingness" may be understood via the container metaphor. This metaphor is erected on the so-called *grounding metaphors* for arithmetic: object collection, object construction, the measuring stick, and motion along a line (Lakoff & Núñez, 2000). Amazingly, their discussion begins with a child's ability to subitize (i.e., recognize small quantities) and culminates with topics plucked from real analysis and complex function theory. In a similar fashion, Núñez et al. (1999) assert that abstract understanding is an outgrowth of physical experiences that are constrained by our biology. This casts light on why students often have difficulties connecting informal and formal ways of knowing. After all, the work of Lakoff and Núñez (2000) in no way supports that *every* formal mathematical notion is metaphorically understood—nor can Núñez et al. (1999) make the analogous claim to human bodily experience.

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[20] In fact, this is one of the main points of their work—that thinking (including mathematical thinking) occurs in the cognitive unconscious.

Summary

This section is broad in scope but aims to convey an idea that has received much attention in advanced mathematical thinking research. It is clear that how we conceive and process mathematical ideas is grounded in our intuitions. For some individuals, informal knowledge may hinder understanding; for others this is the only understanding possible. Still others believe that formal understanding may engender visual embodiments. Although the directional relationship between the two spheres is murky at best, one promising movement is that much attention has been given to forging a connection between the two. Based on this review thus far, it is convenient to synthesize the multitude of principles, theories, and other constructs that seem to have initiated advancements in the extremities of formal and informal thinking, as well as the less-refined continuum in between (see Table 1 on the following page). Some of the constructs in the table are discussed in subsequent portions of this paper but have a clear place in this categorization.

With an eye on defining AMT, one can see the important role of informal ideas (e.g., *concept image* and *spontaneous conception*), formal conceptualizations (e.g., *concept definition* and *vertical mathematizing*), and the bridge connecting the two (e.g., *conceptual metaphor* and *proceptual flexibility*). For these reasons, these constructs are considered influential pieces to defining advanced mathematical thinking and they will be revisited upon formulating a definition.

Informal	Intermediate Phase	Formal
Concept image (Tall & Vinner, 1981)	Structural proof (Leron, 1983, 1985)	Concept definition (Tall & Vinner, 1981)
Spontaneous conception (Cornu, 1991)	Generic organizer (Tall, 1989)	Stipulated definition (Edwards & Ward, 2004)
Extracted definition (Edwards & Ward, 2004)	Cognitive roots (Tall, 1989, 1992)	Vertical mathematizing (Rasmussen et al. 2005)
Horizontal mathematizing (Rasmussen et al. 2005)	Entification principle (Harel & Tall, 1991)	
	Parallel principle (Harel & Tall, 1991)	
	Procept (Gray & Tall, 1994)	
	Conceptual metaphor (Lakoff & Núñez, 2000)	
	Thought experiment (Pinto & Tall, 2002)	

Table 1. Spectrum of Formality

Advanced Mathematical Practice: A Human Activity

A careful examination of the phrase "advanced mathematical thinking" may lead the reader to falsely conclude that this "thinking" is an act that occurs solely in the mind of the learner—one that is immune to external forces (cf. Sfard, 1998). Although portions of this review may appear to support this position, this section frames the individual's developmental struggle as influenced by situational factors including teacher intervention, institutional forces, communities of learning, and other external stimuli. From this point of view, advanced mathematical thinking is seen largely as a composite of thinking and action—a *human* activity in which the individual's thinking cannot be understood apart from acts of engagement. As a consequence of this complexity, this section is divided into two parts: *Instructional Effects* and *Communal Activity*. Following this is a brief summary tying the two.

Instructional Effects

The influence of an authority figure (usually the teacher) has played a central role in shaping research in mathematics education; AMT is no exception (cf. Cornu, 1991; Doyle, 1988). Research that exemplifies this view particularly well is the work of Harel and colleagues (Harel, 1998; Harel & Sowder, 2005). Harel (1998) asserts that appropriate teacher intervention—via intellectual necessity—can foster in students a *way of thinking* which can then prompt a healthy *way of understanding*. The article rests on what he calls the *necessity principle* which avows that students learn best when offered an intellectual need, not a social or economic one (e.g., a good grade). Harel claims that the curiosity and desire to resolve seemingly irreconcilable situations is a large part of what makes us human (Harel, 1998).

By implementing the necessity principle in a unit of linear algebra instruction, the author found that proper channels of understanding were developed by way of students discarding faulty ways of thinking.[21] Hence, ways of understanding can shape new ways of thinking. Coupling this with the results of intellectual necessity, ways of thinking impact ways of knowing and vice versa—precisely the dual assertion on which Harel writes. In Harel & Sowder (2005) there is further discussion on the (sometimes disastrous) consequences of instruction—specifically, how didactical obstacles are formed by embracing instructional myths that trump genuine understanding.[22]

Additional research complements the above work. For example, Doyle (1988) tells us that advanced modes of thinking may not be natural phenomena in mathematics

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[21] Examples of faulty ways of thinking include (a) acceptance of an authority's point of view without question, (b) finding sufficiency in a single representation when additional representations may prove helpful, and (c) reasoning on symbols alone, often masking genuine understanding.
[22] Two such myths are (a) begin with 'simple' and progress to the more complex and (b) use models that are familiar to the learner.

classrooms. That is, students will seek, by direct or indirect means, ways of reducing the ambiguity and risk associated with novel work. This may be interpreted as a consequence of embracing the aforementioned myths of Harel and Sowder (2005). Similarly, Tall (1991b) implores teachers to endorse a change in the norms of classroom teaching. Too often, a teacher's thinking is masked by a polished lecture; hence the arduous task of thinking is hidden and deemphasized. This poisons mathematical growth: "Many mathematicians have learned to present their best face in public, showing their ideas in polished form and concealing the toil and false turnings that littered their growth." (Tall, 1991b, p. 17). This powerful quote illustrates, quite clearly, why many students never reach the level of thinking that we hope. Of course, successful attempts at combating this have been offered (Leron, 1983, 1985) but rarely do these movements gain enough currency to be adopted into state or national curricula recommendations.

A somewhat unique vision in promoting advanced mathematical thinking hinges on the pedagogical tools that can assist in this progression. Specifically, Dubinsky and Tall (1991) discuss two roles that modern computers may play in the development of advanced mathematical thinking. First, sophisticated software such as computer algebra systems may afford opportunities to make connections not possible with pencil and paper alone, so deepening one's grasp of a topic. Second, writing computer programs that emulate mathematical structure (e.g., composition of functions) may assist students in the encapsulation stage of the APOS theory. Although these lines of inquiry are promising, more empirical support is needed.

Artigue (1991), very much in support of such tools, promotes the use of *didactic engineering* in calculus and differential equations—that is, appropriate exploitation of

computer software to develop productive concept images. Such tools ideally permit visualization, experimentation, and generalization from specific examples[23] prior to the instruction of algorithms or algebraic manipulation. This ideology of "play first, operationalize later" has been shown to have a minimal effect on students' ability to perform rote tasks but a noticeable positive effect on students' conceptual understanding (cf. Heid, 1988).

More recent work from Bingolbali and Monaghan (2008) reveals how the personal nature of concept image may be shaped by external factors. In an exploration of the concept images of two groups of students—one group in an engineering program and one group in a mathematics program—the authors found that the engineering students had concept images of the derivative dominated by "rate of change" conceptions while those in mathematics thought mostly of "tangent lines". The researchers argue that these views are outgrowths of (a) teacher privileging and instruction and (b) departmental affiliation and identity (e.g., what it means to be an engineer). This study is one of the first of its kind to connect a well-developed construct in mathematics education (concept image and concept definition) with social theories of learning in an attempt to understand how student conceptions are shaped through teaching and community. As previously discussed, the prior work on concept image/concept definition has focused exclusively on students' individual constructions as somewhat separate from the socio-cultural aspect(s) of learning.

Communal Activity

On par with the adoption of sociocultural and anthropological practices in educational research methodology, it comes as no surprise that recent work in AMT has

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[23] This is what Tall (1989) coined a *generic organizer*.

been shaped by such movements (see Schoenfeld (1992), Cobb (1994, 2007), and Sfard (2008) for more details on these influences). Specifically, Rasmussen et al. (2005) offer a view that not only embraces this shift but seems to hold particular promise (Artigue et al., 2007). The work of Rasmussen et al. (2005) reconceptualizes AMT as more of an *activity* as opposed to a way of thinking. Using the framework of Lave and Wenger (1991), the authors view advanced thought as progressing and evolving through social and cultural activities in the classroom. Put simply, the development of mathematical knowledge occurs through a process of adaptation in which math is both culturally and socially situated. Thus, students must learn both *how* and *when* to do things (Cobb, 1994).

Rasmussen et al. (2005) speak fluently of the duality and exchange of *horizontal mathematizing* (e.g., conjecturing, experimenting, and other informal ways of knowing) and *vertical mathematizing* (e.g., formalizing, justifying, generalizing, and extrapolating). The purpose of horizontal mathematizing is to not only increase the learner's comfort level but to function as preparation for future mathematical activity. They give several illustrative examples through their research in university-level differential equations specific to symbolizing, algorithmatizing and defining. One example provided by the authors discusses how a student's informal response to a question serves as a means of recording and communicating his thinking about population growth. As this informal device takes on new meaning outside of the population context and becomes increasingly general and formal, the learner progresses to a state of vertical mathematizing. In essence, horizontal mathematizing grounds the learner's understanding so that new mathematical realities may be developed (i.e., vertical mathematizing). Here, the

classroom dynamic depicts a reality that is amenable to the thinking that many would characterize as "advanced."

Specific to proof, the social message from Hanna (1991) still resonates today. Here we are told that mathematics is less a game of formality and more about public approval. Hanna asserts that proofs are deemed correct if contemporaries judge their validity and decide it is so. Furthermore, different levels of validity can be achieved so the social nature is woven into the very fabric of creating mathematics. The author mentions that the inexperienced student may falsely interpret clarity and logic as the very essence of mathematics, so missing the point that reasoning and ideas outweigh formal design. If students miss this human element, they may fall victim to positivist traditions, believing that the ageless face of mathematics is "out there" waiting to be found (Thompson, 1992). It is worth mentioning that this work adds credence to the position of Lakoff and Núñez (2000). Grasping mathematics, on their account, is commensurate with grasping the mathematics that *human beings* bring into this world.

Summary

This section explicates the tight bond between the subject of mathematics and the human minds that create it and/or convey its contents to others. It is safe to say that most mathematical experiences will be colored by what happens in a classroom. In fact, it may be that a teacher's portrait of what mathematics *is* will shape students conceptions for years to come (Thompson, 1992). Similarly, the activities in which one engages, irrespective of the nature (rote, exploratory, discovery) or setting (individual, small group, whole class) invariably carries with it social components and social norms. The realization that learning occurs in a social context in which students learn the habits of a

shared community has left its mark on advanced mathematical thinking research. In fact, the culmination of this view is seen in the next section, where AMT is seen to be akin to the thinking done by professional mathematicians.

<div align="center">The Professional Mathematician</div>

A good number of the research papers previously discussed suggest that advanced mathematical thinking is precisely what the practicing mathematician does. This has been a theme that has cut across many reports, whether the focus has been mathematical proof (Alibert & Thomas, 1991; Edwards & Ward, 2004; Selden & Selden, 2003), the developmental struggles in creating new mathematics (Pinto & Tall, 2002; Sfard, 1991), membership into the culture of mathematical activity (Dubinsky & Tall, 1991; Hanna, 1991; Rasmussen et al. 2005; Sfard, 2008), or the flexibility and ambiguity of "dual" modes as suggested by several researchers (Dreyfus, 1991; Gray & Tall, 1994). For the most part, this bond to professional practice and/or *mathematical* research has been anything but a tacit message. With very few exceptions, this has been an organizing theme quite explicit in the work on AMT.

The research domain of proof has received special attention perhaps because of its unique and fundamental place in mathematics (Harel & Sowder, 2007). For example, Alibert and Thomas (1991) favor a setting of scientific debate for learning proof. They contend that by learning the practices of validation and refutation in a debate-like atmosphere, students begin to see these acts as natural tendencies of mathematicians. Specifically, mathematical errors may be viewed as instrumental *sources of reflection* instead of the litter along one's path for neat and tidy results. The researchers also contend that this setting brings individuals closer to scientific autonomy in allowing them

to understand *why* theorems are true. A possible byproduct of these engagements is the coming to terms with why proof is a necessary component of mathematics; this extends far beyond the erroneous perception that proofs are meaningless exercises in formality.

Selden and Selden (2003) write about the findings of a similar study in which students were initially poor judges of correct or incorrect proofs. However, the authors mention that Vygotskian interactions between interviewer and student greatly enhanced the students' judgment and, consequently, positively developed their ability to validate. They use these findings as fuel for research in teaching validation. Along similar lines, the work of Edwards and Ward (2004) conveys the difficulties that most students have in constructing and writing proofs; this challenge is attributed to students' failure to comprehend and employ definitions like professionals in the field.

From a broader perspective, some research supports that the nurturing, revisiting, and eventual maturation of ideas are congruous with the professional mathematician's experience. Sfard (1991) explicates her operational/structural theory through the lens of historical feats and failures and the perpetual struggle to reify. These spontaneous transformations are present in the periodic cycles of mathematical development. Other theoretical contributions (e.g., Pinto & Tall, 2002) posit that professional mathematicians *give meaning* to mathematics; this is in contrast to those who attempt to *extract meaning* from it. In some sense, the practicing mathematician thinks in terms of the wide-ranging applicability of generic strategies—not just deductive reasoning in a box.

With respect to the culture of mathematics and the communities built from engaging in such activity, the previously discussed works of Rasmussen et al. (2005) and Hanna (1991) are a clear fit. Equally important, Dubinsky and Tall (1991) convey their

position regarding computer technology as an emerging enterprise. They make a convincing case for the role of the computer in mathematics education by illustrating what it has recently done for the field of mathematics.[24] The authors remind the reader that mathematicians use computers to experiment, conjecture, organize, test, and even prove important ideas. A student may use a computer in very much the same way even if the mathematics she is studying is already assembled into a well-defined whole. To the *student*, this mathematics is a new investigation, analogous to the practicing mathematician and his quest for new developments.

Gray and Tall (1994) believe that the flexibility afforded by proceptual thinking is something that mathematicians exploit to the fullest extent. They assert that the trained mathematician can detect *when* to unpack the process, object, or intermingled view; this depends on context, experience and/or intellectual need. In a similar fashion, Dreyfus (1991) highlights the importance of representational fluency and the ability to switch between different forms of representation. He frames advanced mathematical thinking processes as extremely complex—ones that include multiple interchanges and junctures of representation and visualization, a switching between complementary forms of representation, alongside generalizing/synthesizing/abstracting from these experiences. He states the following:

> The working mathematician is using many processes in short succession, if not simultaneously, and lets them interact in efficient ways. Our goal should be to bring our students' mathematical thinking as close as possible to that of a working mathematician's. (Dreyfus, 1991, p. 41)

Still others allude to the importance of the acts and mental dealings of practicing mathematicians—that the thinking of these individuals shapes their identity (cf. Artigue,

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[24] Specifically, the computer has paved the way for new areas of study (e.g., chaos and complexity). Moreover, counterexamples may be easily discovered through the use of a computer.

1991; Ervynck, 1991; Robert & Schwarzenberger, 1991; Vinner & Dreyfus, 1989).

Although the connections here are less explicit, it remains clear that the daily practice of

research mathematicians has the potential to reveal essential qualities of advanced

mathematical thinking.

Summary

It might be argued that a person who furiously writes is not engaged in mathematics but a

person who stares at the ceiling is. The well-known mathematician Paul Halmos,

interviewed by Donald Albers in 1990, responded in the following manner when asked

about the best part of being a mathematician:

> What comes first to mind is being alone in a room and thinking . . . I almost always wake up in
> the middle of the night, go to the john, and then go back to bed and spend a half hour thinking,
> not because I decided to think; it just comes. (Ewing & Gehring, 1991, p. 21)

There is little doubt that mathematicians revel in their thinking and consider this the

lifeblood of their profession. Schoenfeld (1992) portrays this idea in framing what he

calls a "mathematical point of view" or a "certain way of thinking" or a "mathematical

way of viewing the world" (Schoenfeld, 1992, p. 344). These ideas capture the essence

of what it means to *be* a mathematician as well as what it means to *do* mathematics, thus

establishing the link between the professional and the student.

Although much of the research gathered here converges in this category of

"thinking like a mathematician" it is this very fact that brings about two elusive qualities

of AMT. First, somewhere along the trajectory of one's career, the naïve mathematician

eventually starts to think like a mathematician and act like a mathematician, and

someday, he *becomes* a mathematician. So one can only conclude that this transition,

whether gradual or spontaneous, is *learned*. A natural question to ask is simply, "How?"

Second, if AMT is really akin to the thinking of professional mathematicians, then Halmos's remark "it just comes," while striking an equal balance of confidence and certainty, is laced with inscrutability. This reveals, if nothing else, the challenges in studying the nature of advanced mathematical thinking.

A SYNTHESIZED DEFINITION OF ADVANCED MATHEMATICAL THINKING

In this section the aim is to construct a rudimentary definition of advanced mathematical thinking—one based on the diverse perspectives reviewed in this paper. It is important for the reader to know that this professed definition is an outgrowth of the literature reviewed, irrespective of personal preference of such issues; this is as much of an attempt at objectivity as one can claim to make. It is in the final section where some constructive criticism will be offered as appropriate. Furthermore, some suggestions will be given as to how the definition can be used to frame new questions that may address some of the shortcomings in the literature.

After much deliberation, a two-part definition was found to be adequate:

Definition

I. Advanced mathematical thinking/tasks/experiences result(s) in a mental reorganization that occurs through a continuous and iterative cycle of knowledge reconstruction.

II. The mathematical experience is conceived as one that embraces a dual existence and functionality across multiple states:

a. *dynamic* vs. *static*

b. *informal* vs. *formal*

c. *in the mind* vs. *in the activity*

Criticisms to each part of the definition are anticipated and encouraged. For example, the first part attempts to characterize AMT by explaining the aftereffect of such a task/experience with this type of thinking. By doing this, one may argue that the writer conveniently avoids having to actually define what AMT is! This is not intentional but is seen to be a widely accepted practice in AMT research.[25] To address this weakness, the second part of the definition is proposed. This offers a glance into the multiple continua that the "advanced" thinker is likely to experience. Furthermore, the reader will also notice the use of the expression "thinking/task/experience" with the conscious intent of embracing the ambiguity that AMT may be "in the mind" of the learner, invoked by a particular task, or fostered through social channels (or more than likely, a blending of these clashing worlds). In the final part of this paper some comments are made that point directly to this dilemma.

Additionally, one might argue that the above definition encompasses such a broad range of phenomena that it is merely a descriptor of *thinking* with no special attention to the subject of mathematics. This would not be the first such remark of this nature. Some time ago, Dubinsky asserted all constructions as "only mathematical in the sense that they have some relationship to standard mathematical concepts, but that, being constructed by the individual, *they are as much mental as mathematical*." (Dubinsky, 1997, p. 72, emphasis added). Time will tell whether the wider research community finds value in such a claim or whether *thinking in mathematics* is, in fact, a vastly different enterprise from thinking in general. For now, support in reference to each part of the definition follows.

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[25] Just a quick glance of the section titled *Criteria for Advanced Mathematical Thinking* confirms this.

Dynamic

Activity

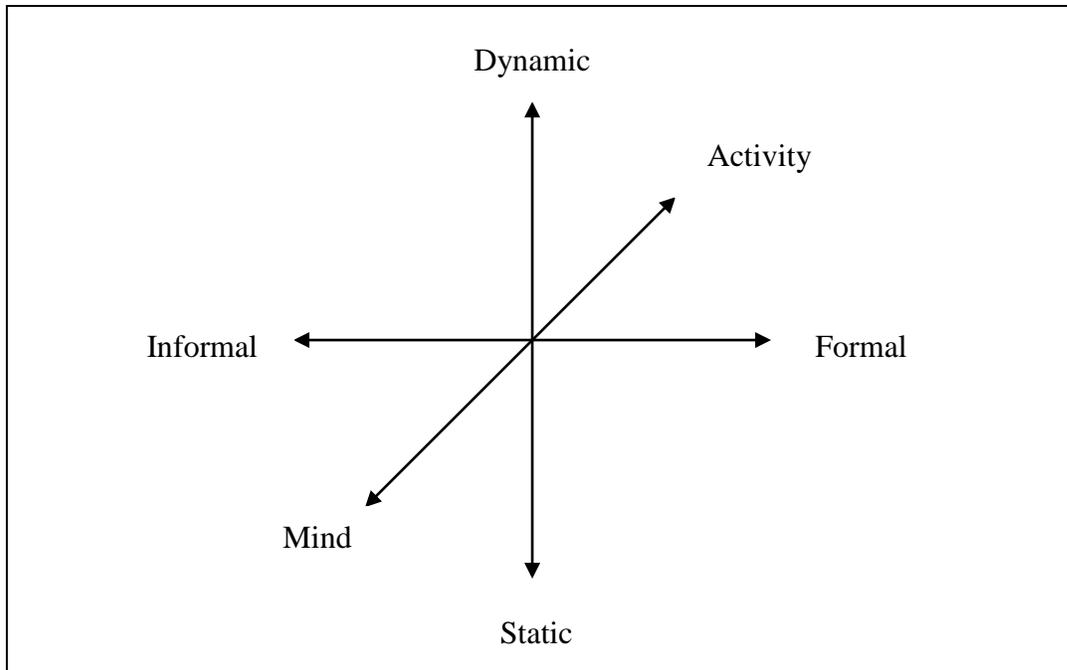Informal ←————————————→ Formal

Mind

Static

Figure 3. Multiple Spectra of Advanced Mathematical Thinking

I. Advanced mathematical thinking/tasks/experiences result(s) in a mental reorganization that occurs through a continuous and iterative cycle of knowledge reconstruction.[26]

The words *reorganization* and *reconstruction* are intentional in the definition. They are meant to imply that the learner has *some* source of information from which to draw, but that this may be inadequate, incomplete, or simply erroneous—so in need of modification. This brings to the fore much of the work discussed in this paper, ranging from the clashing of concept image and concept definition (Tall & Vinner, 1981; Vinner, 1991), overcoming epistemological obstacles (Harel & Sowder, 2005), the dominance of spontaneous conceptions (Cornu, 1991), formulating conceptual blends in our unconsciousness (Lakoff & Núñez, 2000), to the temptation of extracted definitions (Edwards & Ward, 2004). Naturally, as this reconstruction is necessary, the goal is to avoid superficial compartmentalization of concepts (Vinner et al., 1981).

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[26] There is little denying that this statement may be viewed as a recapitulation of Piagetian genetic epistemology.

Iteration and *continuity* are seen as components of knowledge construction in the cognitive theories already discussed (Dubinsky, 1991; Dubinsky & McDonald, 2001; Piaget, 1976; 1978; Sfard, 1991). From these theories, it is clear that new knowledge is created by testing old knowledge in new situations, checking for sufficiency, and renegotiating original conceptions in the event of malfunction. This typifies the iterations and cycles (e.g., the repeated reasoning of Harel (2007)) to which the individual must commit to experience authentic learning. Finally, the spontaneity of encapsulation (Dubinsky, 1991) and reification (Sfard, 1991)—each seen to be analogous to the learner "seeing" something new (Harel & Kaput, 1991; Gray et al., 1999)—are meant to convey that, in fact, some kind of mental reorganization has taken place.

II. <u>The mathematical experience is conceived as one that embraces a dual existence and functionality across multiple states</u>.

Dynamic vs. Static

The ability of the learner to think flexibly and to see the many faces of a mathematical entity has played a dominant role in AMT research. In fact, the difference between success and failure may hinge on one's ability to discern that not all "faces" may be applicable or relevant in a particular context. Gray and Tall's (1994) theory of elementary procepts purports that the learner sees both dynamic processes and static objects through the lens of mathematical representations; Harel and Kaput (1991) assert that mathematical symbolism can capture the ideas of process and object wrapped up in a single notation; Dubinsky and Harel (1991) point out the important skill of toggling between the two.

Sfard's (1991) discussion of operational and structural conceptions—which may be interpreted as dynamic and static modes, respectively—explicates *reification* as the coalescence of the static and dynamic worlds in the sense that dynamic entities are finally seen as independent, manipulable objects. From a different view, Artigue (1991) believes the two can happily coexist by way of generic organizers in a computer setting: "The mathematics is no longer just in the head of the teacher, or statically recorded in a book. It has an external representation on the computer as a dynamic process." (Artigue, 1991, p. 187). The research cited here documents quite clearly that the benefit sits not in having an isolated view of static or dynamic but in seeing (a) both states within a single representation, (b) knowing when to "see" one in favor of the other, and (c) forging a connection and toggling between the two as appropriate.

Informal vs. Formal

There is much talk of intuition and formality, so much that it is difficult to locate a starting point for this discussion. For example, the notions of concept image (Tall & Vinner, 1981), spontaneous conception (Cornu, 1991), extracted definition (Edwards & Ward, 2004), and horizontal mathematizing (Rasmussen et al. 2005) are typically framed as informal phenomenon. Similarly, the notions of concept definition (Tall & Vinner, 1981), stipulated definitions (Edwards & Ward, 2004), and vertical mathematizing (Rasmussen et al. 2005) resonate with the formal side of the spectrum. Where most of the attention has been given (and rightfully so) is in bridging the formal and informal so that the learner may establish connections between the two. These "gray" areas include structural proof (Leron, 1983, 1985), the entification principle (Harel & Tall, 1991), the

parallel principle (Harel & Tall, 1991), cognitive roots (Tall, 1989, 1992), conceptual metaphor (Lakoff & Núñez, 2000), thought experiments (Pinto & Tall, 2002), generic organizers (Tall, 1989), and elementary procepts (Gray & Tall, 1994). Table 1 on page 35 organizes these ideas via this categorization.

It appears that the vast majority of the contributions above support that either (a) AMT facilitates the concretization of abstract notions from personal experience or that (b) the triggering of potential formalities may arise from existing formalities. In fact, Tall (1999) makes an interesting case for each "style" of AMT—first, one that builds on previous work in aiming to develop new knowledge from old and second, one where individuals rely on intuition and experience to guide thinking. Despite the debates on the directionality of such associations, what really matters is that both roads travel to the same destination—the development of new mathematical knowledge by formulation of theorems and their accompanying proofs.

Tall (2001) speaks of the exchange between formal and informal modes of thought (pertaining to infinity) and how natural embodied imagery can emerge from formal constructions. Consequently, these visual embodiments can support future development of sophisticated, formal thinking; this reverses the direction of the prevailing belief that informal thinking shapes the construction of abstract ideas (Lakoff & Núñez, 2000). Similarly, Sfard (1991) states how informal dealings can signify the *culmination* of reifying a process into an object. For example, what is seen today as the conventional number line representation (highly informal) may be the final step in reifying negative numbers as mathematical entities. From this point forward, negative

numbers can be utilized as legitimate objects separate from the process of subtraction (Sfard, 1991).

Gray et al. (1999) comment on different ways of thinking—the *natural* way, built on intuition by giving meaning to the mathematics at hand, and the *formal* way—relying on precise definitions and attempting to understand their significance:

> The cognitive activities involved can differ greatly from one individual to another, including those who build from images and intuitions in the manner of a Poincaré and those more logically oriented to symbolic deduction such as Hermite. (Gray et al., 1999, p. 124)

From this, it is clear that a one-size-fits-all retort for the informal/formal link and a claim as to its directionality are both theoretical dead ends. This sets the table for an emerging debate as to whether the heart of AMT sits in the learner's mind or whether tasks can engender such modes of thinking.

In the Mind vs. In the Activity[27]

Aside from advanced mathematical thinking as a research domain in its own right, a testy debate in education concerns how individuals learn new information (Cobb, 1994, 2007; Ernest, 1995). To illustrate this point, let us momentarily (and artificially) separate two theories, both of which have a very loyal following. On the one hand, constructivist theorists view students as actively constructing meaning via cognitive self-organization (cf. Cobb, 1994; Fosnot, 1996; Steffe & Gale, 1995). This process is adaptive and dependent on how the learner views the world. In turn, one's knowledge is a function of how new information is filtered through the mind, so shaping what is ultimately learned.

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[27] It should be mentioned that the writer sees this as a softer dichotomy than the previous distinctions of *Dynamic vs. Static* and *Informal vs. Formal*. To avoid any unnecessary feuding, one might think about AMT research as mutually benefiting from theorizing *In the Mind* as well as *Looking beyond the Mind*. For example, social constructivism—as a movement in its own right—focuses on the importance of sense-making in a *social* context; this can be interpreted as a blend of what happens inside and outside one's mind.

On the other hand, sociocultural analysts claim that the development of knowledge is through enculturation—a process by which social and cultural situational factors acclimate the learner as to how and when to do specific things (Lave & Wenger, 1991; Vygotsky, 1978). Typically, these views have been in conflict with one another yet both sides have a significant body of empirical support (Cobb, 1994, 2007; Ernest, 1995; Wood, 1995).

Cobb (1994) claims that each view assumes the other but in an implicit, indirect sense. Thus, one could argue for the interdependence of each—that learning through enculturation and guided participation implicitly assumes an actively constructing individual and vice versa. No hierarchy is implied here; instead, Cobb explains that both theories have their merits depending on situative circumstances. From here, one can easily get displaced in the debate as to whether AMT is instinctive to the individual—something that needs to be tended to, if you will—or whether a specific activity or setting can spark one to think in "advanced" ways. If AMT research were to embrace this interdependence wholeheartedly, the community could benefit from what each of the theories brings with respect to research and practice.

To date, the bulk of AMT research seems to be in favor of theorizing on the mind (Cornu, 1991; Dubinsky, 1991; Edwards et al., 2005; Lakoff & Núñez, 2000; Núñez et al., 1999; Pinto & Tall, 2002; Sfard, 1991; Tall, 2001; Tirosh, 1991; Vinner, 1991) or the activity (Alibert & Thomas, 1991; Artigue, 1991; Artigue, Batanero, & Kent, 2007; Dubinsky & Tall, 1991; Hanna, 1991; Harel, 1998; Heid, 1988; Maher & Martino, 1996; Rasmussen et al., 2005) while only a small minority exploits the advantage of a joint view (Harel & Sowder, 2005; Harel & Tall, 1991; Sfard, 2001, 2006, 2008; Tall &

Vinner, 1981). In the next section, the writer argues for the potential role of the *teacher* in fusing the two theories in the context of AMT research.

FUTURE DIRECTIONS

In this section, a brief review of the shortcomings in the AMT literature will be provided as well as visions to address such issues. Not surprisingly, the suggestions proposed are modifications to teaching practice, yet simultaneously offer insights into students' thinking processes. Building from this, the ideal situation may be that of an individual playing the dual role of teacher and researcher to inform advanced mathematical thinking practices. A quarter century ago, Cobb and Steffe (1983) argued that researchers must observe thinking processes firsthand as *teachers*, if in fact, explanations of such processes are desired. The teacher/researcher reciprocity is also important due to (a) the insufficiency of relying on theoretical analyses alone, and (b) the assumption that the learner's *experience* gained through interaction with teachers (not the interaction itself) influences construction of mathematical knowledge (Cobb & Steffe, 1983).[28] This points directly to two weaknesses in AMT research, namely, (a) the research is almost entirely comprised of theoretical findings, and (b) it is often unclear whether the producers of this research are practicing teachers.

Although the research of Cobb and Steffe was conducted with young children, the consideration of such advice may be a wise course of action. The benefits of the researcher/teacher perspective may inform several areas including (a) adding to our understanding of the connection between formal and informal ways of reasoning, (b) implementing ways to fuse this gap, (c) acknowledging and explicating learning obstacles

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[28] Note the theoretical significance of *both* individual construction and situational factors.

associated with AMT, and (d) developing new tools—pedagogical or otherwise—that may result in overcoming such obstacles. Three avenues are proposed here—cultivating concept image, uprooting pedagogical content tools, and establishing cognitive roots. All suggest starting points in instruction which hold particular promise in moving this agenda forward. As the teacher engages students in these informal readiness tasks, the researcher can glimpse into the nature of, and hopefully the progression toward, advanced mathematical thinking.

Cultivate the Concept Image

The motivation for cultivating one's concept image stems from two particular studies even if several research articles make less explicit calls for such action. In studying how students approach mathematical tasks, Vinner (1991) found that most students refer to their concept images *only* in solving problems; definitions are widely ignored. However, it is Vinner's belief that "the ability to construct a formal definition is for us a possible indication of deep understanding" (p. 79). On the other hand, Harel and Kaput (1991) provide empirical evidence that reveals a wide chasm in mathematical thought. Some students rely solely on definitions as a means to methodically verify while others—envisioning the totality of mathematical structures—are able to deduce quite quickly the matters at hand.[29]

On the surface, these two studies may seem world's apart—one stressing the importance of definitions, the other more on conceptual entity. But rather than focusing on this contrasting view, the writer contends there is a tight connection if one were to examine these situations from a *pedagogical* perspective. In both cases, a focus on

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[29] The context here was linear algebra.

shaping the student's concept image is extremely beneficial. In Vinner's case, the spotlight is on enhancement of students' thought habits. That is, one can accept that students use their concept image but taking advantage of this knowledge and shifting instructional attention to this matter is uncommon (see *Pedagogical Content Tools* below). In the case of Harel and Kaput, perhaps more students can conceive of conceptual entities if one makes this a didactic goal (see *Cognitive Roots* below). To date, there are no silver bullets on effective ways to do either, nor will there likely ever be. However, a research agenda that places this theme at the core of AMT studies may prove beneficial in adding knowledge to this research domain.

The crossroads of teacher influence and student thinking—two seemingly disjoint phenomena—is elegantly and concisely intertwined in the remark, "Perhaps enriching students' concept images may be more important than stating definitions precisely." (Selden & Selden, 1993, p. 439). However, Harel and Sowder (2005) are quick to mention the catastrophes of using oversimplified models to do this. In sum, although teaching might initially appear far removed from the agenda of understanding AMT processes, it may be rightfully argued as the centerpiece to gaining insight into this seemingly impenetrable phenomenon. (See Bingolbali & Monaghan (2008) for a specific illustration of this.) Two applications of this view are discussed in the following sections which point to *pedagogical content tools* and *cognitive roots* as fruitful lines of inquiry. The notion of pedagogical content tool is a fairly new development in the research literature whereas cognitive roots have existed for some time but have been explored only minimally.

Pedagogical Content Tools

In reference to the influence that teaching has on student thinking (Cornu, 1991; Doyle, 1988; Harel, 1998; Harel & Sowder, 2005), the work of Rasmussen and Marrongelle (2006) suggests a way to not only avoid common didactical obstacles but foster productive lines of guided thinking. Rather than attempting to "get into the heads" of students, *pedagogical content tools* (PCT)—a blend of teacher intentionality and student thinking with the goal of developing new mathematics knowledge—define a concrete, direct, and tangible way to monitor students' thinking while engaged in classroom practice. To date, two PCTs that have been identified are *transformational record* and *generative alternative*. Transformational records are devices (e.g., notation, a graph, a diagram, etc.) that are used to record student thinking and serve as platforms for subsequent mathematical activity.[30] Generative alternatives occur when the teacher or student suggests alternatives that function in establishing and/or maintaining classroom norms for advancing mathematical activity (Yackel & Cobb, 1996; Rasmussen et al., 2005). For this PCT, the teacher toggles between the "noninterventionist/total responsibility" continuum so that mathematical justification and explanation may profit all.[31] Clearly, these pedagogical tools have the potential to take full advantage of the teacher/researcher reciprocity aforementioned.

In light of the above remarks, transformational records embrace the ambiguity and flexibility of procepts *and* afford opportunities to exploit alternative representations (e.g., in the case of a function—when to use a formula, a table, a graph, etc.). Moreover, given

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[30] I interpret these as *informal* representations of thinking that serve as catalysts for future (often *formal*) developments.

[31] This appears to be a modern-day mathematical whole-class adaptation of the zone of proximal development (Vygotsky, 1978).

the fact that transformational records are *student* generated—thus idiosyncratic and informal except in the most unusual cases—this allows the teacher/researcher to explore the turbulent waters that lead from the concrete to the abstract.[32] In turn, this promotes a fresh perspective on studying this elusive connection so frequently discussed in the AMT research from the 1990s. Additionally, transformational records seamlessly resonate with using concept image as a starting point in instruction.

As Rasmussen and Marrongelle (2006) convey in their work, there are probably many other PCTs that have gone unnoticed. The potential for such tools to inform AMT research and the broader practice of mathematics teaching is enormous. Information from both spheres may foster a healthy atmosphere in which to study mathematical thinking while simultaneously contributing to the research domain of AMT.

Cognitive Roots

Tall (1989, 1992) makes a convincing case for the continued search for *cognitive roots* which may play an important role in curriculum development. Much of the synthesized findings in Tall (1992) reveal that a mathematician's traditional use of definitions, while suitable for building mathematical foundation, is ill-suited to pedagogy. Oftentimes, these definitions—being the finished product of decades (even centuries) of revision—conflict with students' current understanding (or misunderstanding) of the construct on hand.[33] In turn, definitions may be memorized with little to no understanding, planting the seeds for future cognitive disassociations. As Tall asserts, the

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[32] A good example of a transformational record is discussed in the setting of undergraduate differential equations in Rasmussen and Marrongelle (2006). It is reported there that the teacher gathered student reasoning (in a nonevaluative way) about population growth rates by means of small arrows on a graph, indicating local growth. An expert in the field will recognize this as the informal beginnings of the idea of a slope field.

[33] Current research continues to add support to this claim (Paz & Leron, 2009).

product of mathematical thought and the process of mathematical thinking are vastly different but too often, students are provided with the polished artifacts of the former with only minimal exposure to the latter.

Like much of the research presented in this review, Tall (1992) posits that primary sources of difficulty (including concept image/definition disassociation, epistemological and didactic obstacles, spontaneous conceptions, etc.) stem from the complexity of using mathematical definitions as starting points in classroom discussion. He instead advocates *cognitive roots* as launching pads in instruction:

> Rather than deal initially with formal definitions which contain elements unfamiliar to the learner, it is preferable to attempt to find an approach which builds on concepts which have the dual role of being familiar to the students and also provide the basis for later mathematical development." (p. 497).

Two examples of cognitive roots are (a) utilizing "local straightness" to build the idea of limits and differentiation, and (b) using computer programming to assist in encapsulating *processes* into mathematical *objects* (Tall, 1992). Such a starting point in instruction performs several functions including (a) providing a *meaningful* source from which to build (instead of starting with definitions), (b) allowing for expansion rather than reconstruction (presumed to be beneficial as discussed in Harel and Tall (1991)), and (c) maintaining persistence and robustness as to allow more sophisticated ideas to be generated (cf. McGowen, DeMarois & Tall, 2000). It remains to be seen what role cognitive roots may play in the future of mathematics to which there has been minimal application.

CONCLUSION

The common foci in the agenda proposed above are the *learner* and the *teacher*. The research synthesized in this report unanimously supports that learners bring their

knowledge to the table. This knowledge may be problematic across several dimensions—largely intuitive, riddled with misconceptions, or cast with oversimplified ways of viewing the world. On the flip side, sophisticated and creative thinking processes and/or formal deductions may be the norm for some individuals. As the literature attests, no two concept images are alike so choosing a starting point for instruction is a nontrivial task. How can a teacher gain access to this knowledge? If this is even possible, how then can the teacher expand this knowledge?

From the research that has been gathered, the bulk of contributions fall roughly into one of two categories. On the one hand, researchers may observe students engaged in problem solving situations, make inferences about their thinking, and state assertions based on such observations. Equally popular are research reports that are anecdotal in nature—that is, recapitulations of the daily episodes of students' classroom behavior with respect to AMT. The intent of this review is not to criticize either methodological approach; each has contributed immensely to shaping the field. Instead, we wish to make a push for fusing the two approaches to the likes of the research discussed on enriching concept images. Both PCTs and cognitive roots suggest concrete ways in which student thinking can be brought to the fore, discussed in the open, and be put to good use. With all due respect to the research in AMT, it is well-documented that drawing inferences on students in "problem solving mode"—covert by nature—keeps much information hidden from view (and this is an aside from being somewhat unnatural for both student and researcher). Similarly, recounting the impressive tales of daily classroom interactions gives only one perspective—the teacher's.

In moving the above agenda forward, the research outcome is a blend of student thinking and teacher guidance. Not only will teachers be better informed of students' background and readiness but the opportunity to study mathematical thinking as an evolving process in genuine classroom settings is very enticing. It is believed that the community's current understanding of AMT may be greatly enhanced through the lens of whole-class/teacher interaction. Currently, the research on AMT is quite slim in this respect. This is somewhat surprising given the omnipresent push to incorporate more student-centered atmospheres in mathematics classrooms. Aside from this, such findings may add to the existing debate as to the degree of impact of social factors in advanced mathematical thinking.

The future directions discussed here are intended to address these issues and suggest ways that researchers—as teachers—may access, nurture, build, and expand student thinking and knowledge through meaningful classroom experiences. Only then can we even hope to catch a glimpse of what it means to engage in "advanced mathematical thinking." From here, we can take larger and bolder steps forward in helping students achieve this level of thinking in their education and beyond.

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