

Question 2

An individual's epistemological beliefs may be broadly defined as his/her personal credence about the nature of knowledge and knowledge acquisition, usually within a specific domain of knowledge. Based on this definition, how do students' epistemological beliefs about mathematics shape their ways of thinking about and doing mathematics? Furthermore, in what ways do these beliefs have an effect on what is ultimately learned?

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TABLE OF CONTENTS

INTRODUCTION	1
LITERATURE REVIEW	14
Structure of Mathematical Knowledge	14
Certainty of Mathematical Knowledge	24
Speed: Knowledge Acquisition and Problem Completion.....	33
Student and Teacher Roles: Hidden Messages of Classroom Mathematics	38
AN IMPLICIT ASSUMPTION?	51
CONCLUSION.....	52
REFERENCES	58

INTRODUCTION

Students' epistemological beliefs about mathematics has been a topic of interest to educational researchers for decades. This broad area of study is seated at the intersection of two distinct research domains—that of *epistemology* as a philosophical enterprise and *beliefs* in mathematics education. Given this setting, this paper begins with a brief sketch of each—first, epistemology and its roots, followed by a short commentary on the beliefs research in math education.

An assumption that has been somewhat of a mainstay throughout the years is that one's views about the nature, origin, development, and acquisition of knowledge deeply channel one's actions—namely, how one approaches tasks and derives meaning from potential learning situations (Hofer, 2002; Lester, 2002). This position sets the stage for discussing students' epistemological beliefs about mathematics. In order to situate the main findings of this research, a framework fused from the work of Schommer (1990) and Hofer and Pintrich (1997)—to be discussed shortly—will be used as an organizing scheme throughout this paper.

Epistemology

Epistemology is the branch of philosophy that concerns the theory of knowledge (Arner, 1972; Griffiths, 1967; Stroll, 1967).¹ Although some variation can be found in the questions that shape the field, a survey of the abovementioned essays confirms the convergence on a similarly defined nucleus. For example, Griffiths (1967) cites two core questions at the heart of epistemology—specifically, *What do we know?* and *How do we know it?* He states that prior to an examination of these questions, one must study

¹ Some of the outgrowths of epistemology as a *psychological* enterprise are discussed in the section *Knowledge versus Belief*.

knowledge itself. That is, the question *What is knowledge?* should be addressed prior to proposing a theory that claims to dwell on the means of acquisition.

In a similar way, Stroll (1967) argues that the guideposts for epistemology rest in the questions *What is knowledge?* and *How is it possible for one to possess knowledge?* Here it seems that the second question is a blend of the two initially posed by Griffiths. Likewise, Arner (1972) partitions the study of epistemology into an overlapping trilogy of fundamental questions. That is, *What are the limits of knowledge?*, *What are the sources of knowledge?* and *What is the nature of knowledge?* It is this conceptualization in particular that closely aligns with the manner in which a good number of educational researchers (not philosophers) have conducted studies on students' and teachers' beliefs. This view will be a recurring theme throughout this paper.

Within particular domains of knowledge (e.g., reading), some researchers (Cunningham & Fitzgerald, 1996; Fitzgerald & Cunningham, 2002) have developed elaborate frameworks for studying epistemology through the history of philosophical movements. Fitzgerald and Cunningham (2002) see epistemology as concerned with the questions *What counts as knowledge?*, *Where is knowledge located?* and *How do we increase knowledge?* All of these questions can be decomposed on a finer scale revealing issues central to education. For example, a component of the first question might concern specificity versus universality of knowledge. With regard to knowledge locality, one could embrace dualism, monism, or pluralism² (Fitzgerald & Cunningham, 2002). Lastly, do we add to our knowledge base by making sense of situations? Do we discover knowledge or is knowledge created?

² Dualism signifies ambivalence with respect knowledge location; monism to knowledge resting within the process of knowing; pluralism to the multiple locations where knowledge may reside.

The issues above may appear distant from the study of mathematics but the links are undeniably clear. First, specificity and universality can shed light on why students see mathematical problem sets as compartmentalized by specific routines (Garofalo, 1989a; Schoenfeld, 1992). Second, the source and location of knowledge transmit into students' heavy preference for ratification by a teacher or textbook (Diaz-Obando, Plasencia-Cruz, & Solano-Alvarado, 2003; Doyle, 1988; Erlwanger, 1973; Frank, 1988; Lampert, 1990). Finally, we add to our existing knowledge by listening attentively to experts (Frank, 1988; Garofalo, 1989a, 1989b; Schoenfeld, 1988). It is clear that these issues, even if epistemological at the core, personify many of the challenges facing mathematics educators today. Such challenges connect to broad areas of research including studies of classroom culture, student thinking, and pedagogy (Lester, 2007). While on the heels of these issues, it is to the field of mathematics education to which we turn.

Students' Beliefs about Mathematics

Students' beliefs about mathematics has been an attractive area of study for many years, in part due to the surprisingly naïve ideologies that have emerged in the literature. These viewpoints lure students down faulty pathways of reasoning and impede progress toward meaningful understanding. As an example, one might consider asking an algebra student to solve the equation $x^2 + 2x = 6$. Having solved many similar problems in the past, he might choose the typical pathway of first writing $x(x + 2) = 6$ followed by $x = 6$ or $x + 2 = 6$ thus leading to solutions $x = 6$ or $x = 4$. A perceptive student would immediately realize that both "solutions" fail to satisfy the original equation so something is amiss. However, even when provided with this additional information, the original

student might conclude that there is “no solution”—an equally invalid claim. The problem here is that both the solution process and the stated conclusion are highly routinized and deeply entrenched in past experiences where such methods proved error-free.

Of course, several analyses might reasonably illuminate why a student chooses to act in this manner. On the one hand, one cannot overlook that the student might genuinely feel this is a correct way to solve the problem. Alternatively, embracing the belief that mathematics is governed by universal rules for which particular algorithms apply to specific problems, he applies the rules that seem most fitting to the situation and acts accordingly. How can we blame the student for doing mathematics this way? It is likely this is the way he was taught to think about and do mathematics (Greeno, 1991; Lampert, 1990; Schoenfeld, 1992).

Of course, it should be mentioned that this is a hypothetical example in an effort to be provocative. Even so, the research in mathematics education is filled with many such examples suggesting that what students believe about mathematics has a commanding influence on their actions—that is, the reasoning evoked and the processes deployed (Carpenter, Lindquist, Matthews, & Silver, 1983; Greer, Verschaffel, & DeCorte, 2002; Schoenfeld, 1988). Twenty years ago, the *Curriculum and Evaluation Standards for School Mathematics* acknowledged the swaying power of beliefs: “These beliefs exert a powerful influence on students’ evaluation of their own ability, on their willingness to engage in mathematical tasks, and on their ultimate mathematical disposition.” (NCTM, 1989, p. 233). One of the goals of this paper is to elucidate just how powerful this influence can be.

Today, beliefs are widely recognized as being an important component of the affective domain even if some researchers are reluctant to include them there (cf. Philipp (2007) for a recent review on teachers' beliefs). For years, the cognitive and affective domains were treated as isolated systems acting in parallel (Bloom, 1956; Schoenfeld, 1992). With the speculation of interplay between the two domains, theoretical models were devised to consider the complex interchanges between the two. On the cognitive side, Schoenfeld (1983, 1985) contributed much to our understanding of mathematical thinking and sense-making through theoretical developments of how individuals engage in problem solving. His elaborate models revealed several important characteristics of problem solving behavior including the importance of knowledge and resources, allocation and access of these resources, decision-making and monitoring, and the conscious and subconscious beliefs that individuals hold. Meanwhile, McLeod (1989, 1992) reconceptualized the affective domain to include emotions, attitudes, and beliefs—each varying in intensity, stability, and categorization. These research contributions, even with their remarkably dissimilar trajectories, coalesce with *beliefs* which are situated at the juncture of cognition and affect. So while neither specialist may claim “belief” as fundamental territory, one cannot simply ignore its influential power (Schoenfeld, 1992; McLeod, 1994). As Pajares (1992) aptly put it, beliefs are a messy construct.

As evidence of the benefits of the above contributions, much of the research predating this work shares an undesirable byproduct—specifically, the lack of clear definitions of constructs such as *belief* and *attitude*. For example, Schoenfeld (1989) appears to use the expression “students’ perceptions of mathematics” interchangeably with “mathematics attitude” (p. 344); Baroody, Ginsburg, and Waxman (1983) appear to

blend the terms belief and attitude; Underhill (1988) sees belief as an attitude of some sort; Stodolsky, Salk, & Glaessner (1991) use the six words *belief*, *conception*, *perception*, *disposition*, *attitude*, and *view* in a seemingly exchangeable manner. Many additional examples permeate the research literature and serve as sources of criticism for lack of conceptual clarity. Unfortunately, despite the theoretical and conceptual advances previously mentioned, this lack of precision continues to plague the field today (Hofer & Pintrich, 2002; Leder, Pehkonen, & Törner, 2002).

Although there are too many self-constructed definitions in the research literature to reasonably include here, the reader will surely benefit from a *representative* definition of belief, attitude, and emotion (see below).³

Belief

“Students’ mathematics-related beliefs are the implicitly or explicitly held subjective conceptions students hold to be true, that influence their mathematical learning and problem solving.” (Op’t Eynde, DeCorte, & Verschaffel, 2002, p. 24)

Attitude

“A stable, long-lasting, learned predisposition to respond to certain things in a certain way. The concept has a cognitive (belief) aspect, an affective (feeling) aspect, and a conative (action) aspect.” (from Statt, 1990, p. 11 as quoted in Furinghetti & Pehkonen, 2002, p. 41).

Emotion

Emotions or emotional acts are the outward expressions of the beliefs one holds; they are not the beliefs per se. (Carter & Yackel, 1989).

A concrete example may illuminate these differences. For example, if one feels and expresses a particular *emotion* repeatedly (e.g., frustration with integrals) this may eventually shape a fairly stable *attitude* toward integrals (e.g., hatred of integrals).

³ Emotions are included here for the purposes of McLeod’s (1992) reconceptualization. McLeod argues further that there are temporal components to these constructs with respect to stability, intensity, and affective response.

Beliefs, situated on the cognitive end of the scale, take longer to develop and are quite stable (e.g., integrals are really hard). Although such examples are clearly debatable, it is hard to deny that the constructs of belief, attitude, and emotion are indeed different. Treating them as synonyms only adds to their already existing ambiguity.

From the definitions above, beliefs appear most closely tied to knowledge—central to the study of epistemology. Since this paper is concerned with students' beliefs, it is important to flesh out this area a bit more. In the next section the similarities and differences between belief and knowledge will be discussed as a precursor to the framework that will be used to assemble students' epistemological beliefs about mathematics.

Knowledge versus Belief

In this section a telegraphic sketch of the “knowledge versus belief” debate will be provided. Although such contentions date back to the time of Plato, this area of study remains a source of heated debate (Furinghetti & Pehkonen, 2002). While some researchers argue that pointed distinctions only stir the waters and add little to the field (cf. Lester, 2002; Philipp, 2007; Thompson, 1992), mapping out a few of the differences will be helpful for the purpose of establishing coherence and providing a set of referents for students' beliefs about the nature of knowledge.

First, beliefs are often interpreted as *subjective* knowledge—personal and idiosyncratic. That is, beliefs have no truth value (Furinghetti & Pehkonen, 2002) due to lack of consensuality (Thompson, 1992). This contrasts with knowledge being broadly characterized as *objective* knowledge. If a community of experts decides that a particular piece of information is valid, then most would interpret this as knowledge. That is, one

may reasonably assign knowledge the property of being “true” until a refutation of some sort comes along. Consonant with this distinction, DeCorte, Op’t Eynde, and Verschaffel (2002) posit that while beliefs reside in the individual, knowledge rests within a community of knowers.

Another distinction lies in their association—beliefs with doubt or uncertainty and knowledge with truth and certainty (Thompson, 1992). Philipp (2007) argues for defining knowledge as true belief or belief with certainty. Similarly, Ernest (1991) maintains that knowledge is “justified belief” consisting of proposals that are eventually accepted on the grounds of reasonable assertion (Ernest, 1991). This assumption of a knowledge-truth link adds support to the earlier claim that belief contains affective components (McLeod, 1992; Schoenfeld, 1992) while knowledge does not (Furinghetti & Pehkonen, 2002).

The oft cited work of Green (1971) illuminates the uniqueness of belief with respect to *quasi-logicalness*, *psychological centrality*, and *cluster structure*. The first of these asserts that belief systems need not be based on logic (Abelson, 1979). This explains why individuals can hold seemingly contradictory beliefs with little awareness of this difficulty.⁴ On the other hand, knowledge systems are usually grounded in logic as individuals make deductions based on observations or experience (Furinghetti & Pehkonen, 2002; Green, 1971; Kant, 1972).

Psychological centrality emphasizes that while some beliefs are central, others are periphery (Green, 1971). Hence, certain beliefs hold greater weight in the presence of

⁴ Both Schoenfeld (1989) and Op’t Eynde et al. (2002) give powerful justifications for why individuals may hold contradictory beliefs yet see no problems with such a state.

multiple (possibly conflicting) beliefs.⁵ This aligns with the problem solving *personae* as described by Lesh and Zawojewski (2007). Here it is argued that the beliefs or identities of individuals are variable characteristics; one's problem solving *personae* evolves and changes with the situation on hand, thus affording flexibility in unfamiliar situations. This adds merit to the view that contradictory beliefs (as interpreted by an outside investigator) can be held quite happily and unknowingly. On the knowledge side of things, Furinghetti and Pehkonen (2002) assert that “the dimension of psychological centrality is lacking in knowledge systems . . . so if a person *knows* a certain situation, s/he is not prepared to accept a contrasting situation.” (p. 44-45).

It is felt that Green's third component of *cluster structure* may, in fact, reveal similarities between belief and knowledge. The cluster structure of beliefs may explain why one might believe “mathematics is best learned by memorizing” while simultaneously believing that “mathematics helps one to think rationally.” These assertions appear to clash yet a closer examination may reveal that they are, in fact, beliefs about separate entities (e.g., school mathematics versus real-world mathematics). That is, there are clusters of beliefs in certain contexts that need not transfer to other settings. Bogdan (1986) also comments on how beliefs develop in a cluster-like fashion. He asserts that beliefs are not dissimilar from situated knowledge in that students can hold seemingly contradictory beliefs which can be explained by the specific situations that evoke them.

This section concludes with a definition proposed by Philipp (2007)—one that is nearly all-encompassing of the notions previously discussed:

⁵ In fact, beliefs playing a secondary role may appear to dissolve completely to an outside observer.

A conception is a *belief* for an individual if he or she could respect a position that is in disagreement with the conception as reasonable and intelligent, and it is *knowledge* for that individual if he or she could not respect a disagreeing position with the conception as reasonable or intelligent. (Philipp, 2007, p. 267).

On a positive note, the above statement captures the essence of subjectivity versus consensus, doubt versus certainty, and centrality versus peripherality. Additionally, it is clear that beliefs and knowledge cover some overlapping territory. However, Philipp's words also suggest that one person's belief could be another's knowledge. In effect, this refocuses the question back to the origin and location of knowledge.

If beliefs and knowledge are interchangeable constructs, perhaps we would answer the questions *How do you know that?* and *Why do you believe that?* in identical ways. However, the first question hints at knowledge structure or the way(s) in which individuals organize understanding with accompanying warrant. Meanwhile, the second question requires a personal defense (or so it seems), one that is vulnerable to outside criticism.⁶ As the book of "knowledge versus belief" continues to be written, it is clear that (despite the minor disagreements) reasonable progress has been made on shaping their many similarities and differences.

An Organizing Scheme for Students' Epistemological Beliefs about Mathematics

In response to the scores of conflicting characterizations put forth in defining *epistemological belief*, Hofer and Pintrich (1997) state that "it is sometimes unclear to what degree researchers are discussing the same intellectual territory." (p. 111). Despite this fact, an abundance of the definitions share the common thread of discriminating between beliefs about (a) the nature of *knowledge* (e.g., in a particular domain) and (b) the processes of *knowing* (DeCorte et al., 2002; Diaz-Obando et al., 2003; Hammer,

⁶ Again this supports the view that belief is highly personal.

1994; Hofer, 2002; Hofer & Pintrich, 1997; Leder et al., 2002; Schommer-Aikins, 2002). Amidst the confusion, what remains clear is that the first component addresses beliefs about subject matter as a (supposedly) well-defined body of information while the second concerns the acquisition of this information.

Hammer's (1994) definition of epistemological beliefs in physics, amidst others, typifies this duality well. He states that epistemological beliefs are "beliefs about what constitutes knowledge in physics and how, as a student, one develops that knowledge." (p. 152). This definition clearly extends to subject areas beyond physics. Given the straightforward nature of this definition and the fact that it resonates with the two prongs above, it is this definition that will be taken as representative of *epistemological belief* for this paper.⁷ The specific contents of the two prongs are elaborated in the paragraphs that follow.

The question *What is knowledge?* or more specifically, *What is mathematics?* is fundamental to personal epistemology (see earlier discussions). In general, attempts to inform this area are anchored in students' interpretations of the stability and certainty of mathematics. Additionally, student views on mathematical structure (i.e., how the subject is assembled, disassembled, and subsequently applied) has been a dominant theme in mathematics education. Questions concerning these areas often point to student beliefs about the organization, coherence, structure, and stability of the subject. Beliefs from the literature going back some 25 years are summarized in the two sections of this paper *Structure of Mathematical Knowledge* and *Certainty of Mathematical Knowledge*.

⁷ It should be stressed that this choice is rather arbitrary since other definitions captured this duality equally well. However, this one was chosen specifically for its lack of unnecessary jargon.

The question *How does one learn mathematics?* addresses the second area of the framework. As above, it is further divided into sections—mainly, source(s) of knowledge and justification of knowledge. Source(s) are associated with the expectation(s) of both (a) authority (textbooks and teachers) and (b) the self as learner of mathematics. Naturally, reflections of self spark debate with issues such as accessibility, speed, and control. Finally, justification is primarily concerned with how one evaluates claims, puts evidence to use, and asserts truth. As will be seen shortly, these actions are often deeply affected by classroom teaching and other “environmental” factors. The sections *Speed: Knowledge Acquisition and Problem Completion* and *Student and Teacher Roles: Hidden Messages of Classroom Mathematics* address these concerns. The heart of this paper attends to each aspect of the framework as students’ thinking and actions are seen to be steered by their beliefs about mathematics.

For the reader’s convenience, this organizing framework is summarized in the table below.

Knowledge	What is knowledge in mathematics?
<i>Structure</i>	Benny believed that there were rules for different types of fractions... (Erlwanger, 1973, p. 10).
<i>Certainty</i>	...mathematical truth is corrigible, and can never be regarded as being above revision and correction. (Ernest, 1991, p. 3).
<i>Simplicity</i>	Almost all mathematics problems can be solved by the direct application of the facts, rules, formulas, and procedures... (Garofalo, 1989a, p. 502).
Knowing	How is knowledge acquired and demonstrated?
<i>Speed</i>	Students who understand the subject matter can solve assigned mathematics problems in five minutes or less. (Schoenfeld, 1988, p. 151).
<i>Lessons learned from classroom experience</i>	...at school the implicit goal of learning is to please the teacher. (Diaz-Obando et al., 2003).
<i>Source</i>	Many students believed that only the teacher could tell them whether an answer was right or wrong. (Frank, 1988, p. 33).

<i>Justification</i>	...the students had learned to regard themselves as a mathematical community of discourse, capable of ascertaining the legitimacy of any member's assertions using a mathematical form of argument. (Lampert, 1990, p. 42).
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Table 1. Organizing scheme for students' epistemological beliefs about mathematics

Under the broad headings of *knowledge* and *knowing*, sample quotations representative of one side of the debate are provided. The roots of this organizing scheme can be found in Schommer (1990) and Hofer and Pintrich (1997). On the one hand, Schommer uses the five independent categories of *stability*, *structure*, *origin*, *speed*, and *control* to study students' epistemological beliefs. In contrast, Hofer and Pintrich draw a line distinguishing *knowledge* from *knowing*; given this demarcation, *certainty* and *simplicity* define knowledge while *source* and *justification* comprise knowing. Since there is merit in each of these perspectives, the framework used in this paper will fuse the two by considering four of Schommer's dimensions situated across the knowledge/knowing dichotomy articulated by Hofer and Pintrich. Furthermore, additional areas will be explored (even if subordinate to these general categories) as they prove influential in shaping students' ways of thinking about and doing mathematics.

It should be mentioned that the categories developed for this paper are fitting inasmuch as they reveal specialized beliefs that students hold about particular areas of mathematical knowledge and knowing. However, with any organizational structure comes the limitation of too narrow a scope and/or too artificial a compartmentalization. For example, the reader may find that some of the beliefs—due to their complexity and connectivity to many classroom factors—may be aptly placed into multiple categories. In fact, this is the reality and is reflected in the frequent cross-referencing to other areas

of this paper. Additionally, since many research reports contain findings that fall into more than one category, the treatment of these papers will be very detailed upon their introduction but then much less so with a second or third mention. This is done not only to conserve space but to avoid repetition.

LITERATURE REVIEW

The organization for the remainder of this paper is as follows. First, attention to the relevant research will be given under the categories previously developed—*Structure of Mathematical Knowledge*, *Certainty of Mathematical Knowledge*, *Speed: Knowledge Acquisition and Problem Completion*, and *Student and Teacher Roles: Hidden Messages of Classroom Mathematics*. From the research gathered, the beliefs will be identified and discussed in the main body and then briefly summarized for convenience. Following this, a discussion of how these beliefs infiltrate students' thinking and actions will be offered.

This section begins with the study of how students perceive the ways in which mathematics is organized as a discipline in its own right. Addressing this topic initially is a logical starting point since these beliefs provide the underpinnings of many additional beliefs discussed later in the paper.

Structure of Mathematical Knowledge

What one thinks mathematics is will shape the kinds of mathematical environments one creates, and thus the kinds of mathematical understandings that one's students will develop.

—Alan Schoenfeld, 1992, p. 341

This section discusses the beliefs that students presumably hold about the subject of mathematics. Central to this category is the question *What is mathematics?* Hofer and Pintrich (1997) use the term *simplicity* with regard to this category while Schommer (1990) and Hammer (1994) uses the term *structure*. Regardless of the label chosen, the general consensus is that students view mathematical knowledge in one of two ways—either as disparate facts lacking cohesion or as an integrated body of information, rich in patterns and connections. The former view may instill in students a pointillist approach to the study of mathematics, one in which instrumental understanding is a top priority. On the other hand, an integrated perspective may promote deeper understanding of relationships and concepts (Skemp, 1971). Naturally, a student may not feel strongly one way or the other and may conceivably choose “middle” ground. However, a healthy portion of the research confirms beliefs consonant with the former view.

A good starting point is the case study of sixth-grader Benny (Erlwanger, 1973). This oft cited work criticizes a program in mathematics called IPI (Individually Prescribed Instruction) by revealing the harm inflicted in shaping a child’s conception of mathematics. Specifically, Benny has found efficient yet peculiar ways of succeeding in school mathematics. He devises patterns of overgeneralization based on shamefully faulty reasoning in his quest to understand fractions and decimals. For example, he believes that the number .5 can be expressed as a/b , where $a + b = 5$ (so, e.g., $4/1=.5$, $1/4=.5$, $2/3=.5$, etc.); although this procedure often leads to incorrect answers, the frequent contradictions do not bother him. He believes there are hundreds of individual rules for dealing with specific problems and that such rules were invented by intelligent minds.

The author lambastes the IPI program for its overly simplistic behaviorist structure. By having students “master” the skills and achieve an 80% benchmark, Benny is just one example of what can happen when meaningful understanding is ignored. Erlwanger asserts that Benny’s experience with IPI (since second grade) has fostered unhealthy views about mathematics. These assertions were made based on a series of semi-structured interviews in which the author formulated new questions from Benny’s responses. Outwardly disturbing is Benny’s admission that mathematics is a “wild goose chase”—specifically, this is the reference Benny makes to trying to obtain the exact answers that appear on a key which is used to determine one’s progress. Even more shocking, Benny equates mathematical answers and rules with *magic*. He explains, “I am going to look up fractions, and I am going to find out who did the rules, and how they are kept.” (p. 19). In short, Erlwanger’s research—being one of the first of its kind—serves as a reminder of an appalling program failure, one which discounts mathematical reasoning and understanding to such a drastic degree that a student would devise alternate (nonmathematical) pathways to success.

The work of Garofalo (1989a, 1989b) paints a picture similar to Erlwanger’s Benny but with a notably wider audience of students (ranging from elementary school through college) as well as pre-service/in-service teachers of mathematics. Unfortunately, two major drawbacks of this work are its anecdotal flavor and lack of detail, thus making it susceptible to a variety of interpretations. Limitations aside, the accounts told are powerful, palpable, and often disturbing. Moreover, the findings mirror that of more systematic research studies (cf. Schoenfeld, 1988, 1989) hence adding somewhat to its credibility.

Through the use of informal interviews, classroom observations, and casual conversations with students and teachers, several classroom episodes are described with unusual clarity. For example, Garofalo (1989b) details the struggles of an elementary school mathematics class attempting to solve the problem $3 - 2\frac{1}{2}$. The teacher was leading the discussion at the board while the investigator sat at the back of the classroom. As the discussion began, students were eager and beaming with ideas. However, all suggestions alluded to a procedure or routine that could be easily automated: “bring down the $\frac{1}{2}$,” “subtract the 0 from the $\frac{1}{2}$,” etc. (p. 452). Garofalo argues that such suggestions emanate from the belief that rules and procedures are hallmarks to mathematical thought. More recent studies (cf. Diaz-Obando et al., 2003; Higgins, 1997) confirm this narrow view of mathematics as the widely accepted custom.

At about the same time, Frank (1988) reported on a study in which four high ability middle school students were interviewed a minimum of four times, each with respect to mathematics and mathematical problem solving.⁸ Although minimal information about the study’s instrument(s) is provided, the beliefs discussed pervade the literature in mathematics education. The format of data collection included direct questioning about mathematics, classroom observations, analysis of think-aloud protocols, and completion of a survey.⁹ The prevailing beliefs that emerged were (a) math problems typically have a single correct answer and (b) memorization is important. There is a synergy in viewing these beliefs with respect to the rule-based view just discussed. That is, if one sees mathematics as overtly procedural, then one had better *know* these procedures (perhaps by memorizing them). Moreover, if one carries out a

⁸ Their ability level was evidenced by their enrollment in an intensive fast-track program for talented students.

⁹ None of this information is provided for the reader but it can be found in Frank (1985).

procedure, there is no room for error—the solution is either correct or incorrect. The beliefs cited in Frank (1988) are confirmed in Spangler (1992) in a surprising consistency across many grade levels (elementary, junior-high, senior-high, and graduate level). Resonating with Benny’s view as well as cutting across the work of Frank and Garofalo, Schoenfeld (1992) adds that students typically see problem sets as being associated with specific methods, further supporting the view that mathematics is best learned in bits and pieces.

The *National Assessment of Educational Progress* (Dossey, Mullis, Lindquist et al., 1988) gives further evidence of this defective view as embraced by high school students. There it is reported that more than 80% of eleventh graders believe rules governed mathematical tasks. Similarly, the national report *Everybody Counts* found that almost half of eleventh graders see mathematics as mostly memorization (NRC, 1989). Schoenfeld (1989) found analogous results in the responses to the following question given to high school students: *How important is memorizing in learning mathematics? If anything else is important, please explain how.* Responses such as “Memorization of equations and formulas are essential in mathematics” and “Without knowing these rules, you cannot successfully solve a problem” were representative of the majority (p. 344). Clearly, these results converge to paint an alarming portrait of mathematics education at the time.

The work of Schoenfeld (1985, 1988, 1989) was influential in both igniting an interest in beliefs research and paving the way for future studies. Specifically, Schoenfeld (1988) gives a powerful account of the unintended messages that students receive during mathematics instruction. The beliefs previously cited in this section are

clearly present in his work but additional elements not often found in other studies also emerge. Specifically, students' artificial separation of mathematics into convenient pockets of isolation makes its appearance here. Both Cobb (1986) and Schoenfeld contend that this polarization makes the subject easier to manage—so functioning as a coping mechanism in some sense. Two of these “separation” beliefs are omnipresent in Schoenfeld's work:

- (a) Mathematical theory/proof are unrelated to exploration/discovery.
- (b) Mathematics in school is different from the work of professional mathematicians.

Each of these is discussed in greater detail below.

Schoenfeld's (1988) study examines the mathematical beliefs of tenth grade geometry students in upstate New York. Through a year-long study that included classroom observations, questionnaires, and interviews with students and teachers, he found that students in various classes viewed deductive geometry (two column proofs) and constructive geometry (with compass in hand) as detached branches of mathematics. For example, “proofs were seen as the formal confirmation of results that are already known” (Schoenfeld, 1989, p. 341). This resulted as a consequence of students believing that the final assessment of a geometric construction is that it “looked right.” In sum, proofs were formal, codified exercises in the (re)validation of pre-established truths—hardly a tool with potential to invent new mathematics. He once called this the dichotomy of empiricism versus deduction (Schoenfeld, 1985) and has frequently pointed to diSessa's work in naïve physics as a launching pad (cf. diSessa (1993) for details).

Schoenfeld's (1989) work examines how students' conceptions of mathematics shape their behavior in mathematical engagements. This is a fitting piece of research that provides "middle ground" data as a supplement to the fine-grained qualitative findings in Schoenfeld (1988) and the large-scale assessment data from NAEP (Carpenter et al., 1983).¹⁰ To a degree, this work substantiates much of the anecdotal work of the time in providing corroboration as well as raising new questions.

Two-hundred thirty high school students in upstate New York were the subjects; all were enrolled in college bound mathematics. The instrument used in this study contained 81 questions (70 using a four-point Likert scale, 11 free response). The items address a variety of areas including causal attribution, views about mathematics and classroom practices, motivation, views across different subject areas (math, social studies, English), and relationships between formal and constructive geometries. Validity and reliability of the instrument are not discussed in the article but face validity is evident from viewing the instrument.

Amidst the findings already echoed here, students' compartmentalization of mathematics learned in school and mathematics done by professionals is evident (much more on this later). This explains why one can hold the seemingly conflicting views that (a) mathematics helps one to reason and think creatively (very much the case for professional mathematicians), versus (b) memorization is important to succeed in mathematics (as might be the case for students experiencing math in a school setting). Schoenfeld argues that this splintered view supports the assertion that students successfully cope with the constraints present in school settings (see also Cobb (1986)).

¹⁰ Moreover, it appears that the results from Schoenfeld (1989) are drawn from the companion works of Schoenfeld (1985, 1988).

In fact, students may espouse to believe a variety of claims, but when it comes time to “doing” mathematics and meeting individual goals, students will almost certainly rely on whatever has worked in previous situations. The mantra “do what you need to do” typifies the separation of proof/invention from school/real-life; in many cases, memorizing facts and applying algorithms function quite well. This behavior can regulate itself over time promoting an artificially narrow approach to studying mathematics. Naturally, the end result presents students with significant barriers when faced with future mathematical experiences (Schoenfeld, 1985, 1988).

Summary: Structure of Mathematical Knowledge

In this section, mathematics is seen by many as a subject governed by universal rules in which procedures and algorithms dominate the landscape. Consonant with this view, some students may come to believe that mathematics has to be done in a prescribed way, leading all persons to the same result and fostering the belief that math problems having a single correct answer. If remembering rules is important, then thinking translates to recalling memorized facts, thus relegating derivation and reasoning to diminished levels. This escalates into the view that formal mathematics and proof are games of some sort—games that one should play in school but not in daily life.

Discussion of the Effects of Beliefs: Structure

Although some of the findings above are not based on research with strong drawing power, there is a good portion of large scale assessment data that supports this anecdotal work (NAEP, 1988; NRC, 1989). From the studies briefly outlined here, student *thinking* is often seen as remembering the correct procedures. Garofalo (1989b)

adds that students falsely interpret *understanding* mathematics as being able to follow these procedures. From this, one might surmise that students prematurely decide that they understand if they “get it” in class. This may explain why students are frequently baffled once they try to engage in independent practice at home. In the end, there is little debate that a large number of students approach mathematics in a very mechanical way, even if the exact proportion is unknown (Garofalo, 1989a, 1989b; Higgins, 1997).

The belief that formal mathematics is unrelated to discovery implies a specific way of thinking about mathematics. That is, students may interpret mathematics as a closed system in which memorization is not only plausible but the most reasonable way of acquiring “understanding”. Hence thinking translates to rehearsal as detailed above. This, in turn, connects to one’s way of *doing* mathematics. In fact, Schoenfeld (1983) gives a lucid example of this in a protocol in which two students attempt (but fail) to devise a geometric construction. Their various maneuvers are mostly based on empirical conjectures, what “looks” right, occasional leaps of desperation, and trial and error. However, at a later point (after videotaping) the investigator asked the two students to solve two additional problems (both geometric proofs) and they did so in less than five minutes each. The surprising outcome here is that the latter two problems comprised the *solution* to the problem on which they initially struggled! In other words, the students possessed the mathematical sophistication to solve the original problem but they simply did not bring these resources to bear.¹¹

Schoenfeld claims that the above behavior is largely a consequence of the belief that proofs act merely as outlets to verify information already categorized as “true.” He

¹¹ This article also addresses the “environmental” factors of videotaping as inducing pathological behaviors in the participants.

calls this the “implicit rejection of proof” since even if unconsciously held, it presents a powerful force in one’s problem solving behavior: “One’s beliefs establish the context within which one (a) selects from among his ‘resources,’ and (b) employs them.” (Schoenfeld, 1983, p. 346). His careful use of language suggests that this belief influences not only how students think in problem solving situations but also the actions they carry out while doing mathematics. To appease any skeptics, Schoenfeld compares this episode to that of a professional mathematician’s work. Although the mathematician had not worked with geometric ideas for over a decade, he agreed to attempt a construction problem similar to what was asked of the students. The fundamental distinction was the mathematician’s use of proof as a means to complete the construction. In fact, proof was so essential to his solution that he asked the investigator if completing the construction was even necessary.

The research in this section documents convincingly that students almost exclusively “do” mathematics by applying the standard algorithms taught. Implicit in these actions is that mastery of mathematics is equivalent to procedural fluency of these fragmented concepts. As previously discussed, Garofalo’s (1989b) report on the problem $3 - 2\frac{1}{2}$ supports this mechanistic view with such severity that one could conceivably argue that students suspend the use of common sense in lieu of previously learned procedures (Greer et al., 2002; Verschaffel & DeCorte, 1997). The bottom line to much of Schoenfeld’s work is that this extreme duress is readily observable in our mathematics classrooms as a result of years of classroom conditioning.

The consequences of these misinterpreted views are as plentiful as they are discouraging. In light of the above discussion, they include:

- (a) placing a premium on instrumental understanding and/or procedural fluency, usually via following rules,
- (b) suspension of thinking and reasoning in lieu of memorization, so diminishing the importance of derivation, and
- (c) drawing arbitrary and divisive lines across the subject, that is,
 - a. school mathematics vs. professional mathematics
 - b. formal mathematics vs. discovery.

The above notions favor a view of mathematics as fragmented and disconnected.

Although such views may prove debilitating in future encounters with the subject, these beliefs have proven themselves effective in school settings. For example, embarking on “wild goose chases”—as ludicrous as this might appear—becomes somewhat of a ritual to mathematical activity. As the reader shall see shortly, several of the beliefs embraced here connect to students’ beliefs regarding the degree of stability in mathematics. The frequently adopted view of mathematics as an immutable and ageless science is the area that we turn to next.

Certainty of Mathematical Knowledge

Commonly, mathematics is associated with certainty: knowing it, with being able to get the right answer, quickly.

—Magdalene Lampert, 1990, p. 32

This section aims to discuss student beliefs concerning the stability of mathematical content. Like the previous category, personal views on the degree of certainty are well-recognized as an important component of epistemological stance (Hofer, 2002; Muis, 2004). In order to measure this type of belief, Schommer (1990) uses the term *stability* with “fixed” and “tentative” representing opposite poles; Hofer and

Pintrich (1997) choose the term *certainty* with “absolute” and “malleable” as descriptors. Rightfully so, much of the research cited in the section *Structure of Mathematical Knowledge* intermingles seamlessly with these beliefs. It can be argued that how one mentally configures and organizes the subject matter (structure) and what one claims to be mathematically valid (certainty) have inextricable connections. This section begins by reflecting on a personal experience (more than 20 years past) to make this point.

An example from my seventh grade social studies class illustrates how the areas of *structure* and *certainty* may connect in unexpected ways. The teacher had asked that the class come prepared the next day to write down the names of all of the U.S. Presidents in the order of service, from George Washington to Ronald Reagan. She boldly claimed that this would help the class remember the important events in U.S. history in connection with the country’s greatest leaders and thinkers. We were given the remainder of the period to begin “studying.”

My classmates began the arduous task of connecting the Presidents’ names via the influential events of the time—rebellions, revolutions, slavery, suffrage, World Wars, Watergate, and the like. I, on the other hand, saw the task in a completely different way. After all, the assessment we were to face the very next day—being as contrived as it was—was to simply write down the names of the Presidents. Specifically, I saw “George Washington” as an isolated bit of information—one to be memorized and stored for later retrieval (the test). In fact, the single letter W could be used as a code for Washington, minimizing the load even further. Before I knew it, the task became memorizing the string WAJMM... That evening I wrote a poem that began as such:

With Aunt Jemima my mother
 Always juggled very hot pancakes.
 Twenty-five pancakes—blueberry, lemon, jelly . . .

Given the narrow and absolute nature of the task, it afforded me the opportunity to structure the content as disparate pieces of information. I then reconnected the pieces via a mnemonic device which produced, for me, the desired outcome—a good grade. I remember feeling guilty since my “success” was a product of intentionally deflecting the teacher’s good intentions.¹² But this example hardly seems different from Benny’s effort to develop a wide array of disconnected rules to be successful with decimals and fractions. In fact, one might go as far as applauding Benny for his efforts given that he had little to no foundation from which to develop his understanding. He invented rules much like I invented the poem.

A fitting study here is the work of Stodolsky et al. (1991) in revealing the effects and origins of beliefs in mathematics and social studies. The two disciplines were chosen precisely because of their contrasting nature with respect to well-definedness, “typical” methods of instruction, and level of student activity. Given these assumptions, this study describes the differences in student views across the two disciplines with an emphasis on the stability of subject matter content.

The researchers interviewed 60 fifth graders spanning two years. A total of 11 classrooms across six Midwestern schools were involved and students were chosen based on a random selection of those children whose parents returned permission slips for interviews. Interviewers were trained graduate students and children’s responses were coded by two researchers.¹³ The areas that were probed in this study were (a) *definition* (How do these children define social studies and math? What is knowledge in

¹² Despite the poem, one could argue that I still obtained the type of understanding the teacher desired.

¹³ It is mentioned that 91% reliability was achieved when four random protocols were analyzed for coding consistencies.

math/social studies?), (b) *classroom norms* (What are typical classroom activities?), (c) *learning* (How do these children perceive learning the content? How does one acquire the knowledge of math/social studies?), and (d) *accessibility* (Could you learn this content on your own?).

Before the interviews were conducted, the researchers visited and observed each class to gain a sense of classroom regularities and norms. As discussed in their report, the math classes were traditional in that lectures and independent seatwork were dominant. By contrast, the social studies classes were so diverse that it was difficult to make a generalization (e.g., topics/activities ranged from analyses of current events, map making, journal writing, role playing, trips to the library, reading from a text, etc.).

With regard to stability in the subject area of mathematics, Stodolsky et al. (1991) report on students' views of mathematics as static and unchanging. In a chi-square analysis, the 60 students conceived math content as less malleable than in social studies, suggesting different statistical distributions with respect to content area ($\chi^2 = 10.34$, $df = 4$, $p < 0.05$) (p. 108). This was in response to the question "Can you think of something to change?" Although some respondents answered "yes" to this question with respect to math, the comments were not directed toward content per se. Student comments in social studies strongly suggested freedom to select different areas of coverage (including the depth of coverage) whereas remarks in mathematics suggested improvements in time management, reducing boredom, and other administrative issues. In short, the *implicitly defined core* in mathematics was perceived as immutable: "Math is conceived to be particularly unchangeable in terms of actual content . . . no one

questioned that math could be about anything different than what they cover in school.” (Stodolsky et al., 1991, p. 109).

As much as the previous section illustrates, the responses and reactions of these students align with their beliefs (Schoenfeld, 1985, 1988, 1989). Furthermore, the unchanging nature of mathematical content (as perceived by many) may be a key determinant in why students choose to master its contents by memorization (Frank, 1988; NRC, 1989; Schoenfeld, 1989; Spangler, 1992). If math (or any other content for that matter) remains stable from day to day, memorization is an efficient and attractive way to internalize new information, as was the case of my seventh-grade experience.

The work of Lampert (1990) adds support to the prevailing view of mathematical certainty by describing a fifth grade class’s struggle to change its views on the nature of mathematics. She offers an interesting perspective that relates to *belief change* although some of the findings are difficult to substantiate since the reader is offered merely a taste of what happened in this classroom.¹⁴ Borrowing heavily from the mathematical perspectives of Lakatos (1976) and Polya (1954), Lampert sets out to discuss how the teacher creates and sustains a culture that redefines the meaning of *doing* mathematics to align with the authentic nature of the discipline. A classroom episode is examined and interpreted from a mathematical, pedagogical, and anthropological (sociolinguistic) lens, offering a rich and informative portrait of classroom practice. In creating an atmosphere where argumentation is the norm, she discusses freely the vulnerabilities of student risk-taking as seen through public displays of thinking as well as the novel aspect of taking charge of one’s thinking. It is here where she refers to the traits of *intellectual courage*,

¹⁴ Although belief change is not a central theme of this paper, Lampert’s description of students’ struggles conveys the certainty of mathematics *indirectly* (i.e., not by use of questionnaires and/or interviews).

intellectual honesty and *wise restraint* (Polya, 1954) that are so critical to making this engagement successful for all.

In short, the teaching episode provides evidence that it is possible to help students redefine what it means to *know* mathematics through the teacher proactively reshaping what it means to *do* mathematics.¹⁵ Creating an environment that welcomes hypotheses accompanied by public defenses, many of the ill views so prevalent in mathematics did not arise here. Lampert provides detailed vignettes of students making conjectures and assertions through experimentation and inductive reasoning. While defending such views, students eventually shifted to deductive chains of command and ultimately, formal and informal proof. This, in turn, provided fertile ground for new conjectures to develop. The strength of this paper is its ability to illustrate the widespread acceptance of the immutable nature of mathematics by way of illuminating the struggles (but ultimate success) of a class attempting to *amend* this view.

Finally, the work of those studying prospective teachers' beliefs about subject matter mirrors much of the work seen here. On the one hand, this finding is surprising given the hope that those in the classroom would embrace more progressive views about their subject matter. However, pre-service teachers—being students of mathematics themselves—may harbor the same views discussed here (Ball, 1990; Frank, 1988; Garofalo, 1989a, 1989b). On the other hand, this is highly anticipated given the wealth of evidence that teachers (a) generally teach the way they were taught (Ball, Lubienski, & Mewborn, 2001; Shulman, 1986), (b) may have considerable difficulties in teaching for conceptual understanding by way of a hidden propensity for rule-based mathematics

¹⁵ The use of the prefix “re” in *redefining* and *reshaping* is subtle but important—suggesting that students arrive with default views about how mathematics should be perceived.

(Ball et al., 2001; Eisenhart, Borko, Underhill et al., 1993; Even, 1993), or (c) may generally have a weak grasp on the theoretical foundations of such widely used algorithms/procedures (Simon, 1993; Simon & Blume, 1994). Although the work of these researchers is clearly important for mathematics education, it sits beyond the immediate scope of this paper.

Summary: Certainty of Mathematical Knowledge

Certainty is central to mathematics—knowing something in mathematics is consonant with generating quick and accurate solutions (Ball, 1990; Lampert, 1990; Schoenfeld, 1985). Along with this belief comes the safety net that mathematical knowledge has a very long shelf life—knowing math today means you will know it tomorrow. Such a belief in the unconditional nature of mathematics nurtures a prescriptive regiment for learning—one in which old-fashioned thinking is usurped by exacting procedures (Schoenfeld, 1985). It is this pervasive belief in the characteristic of mathematics that propelled reform movements in the 1960s and 1970s to a shift of “doing mathematics as an act of sense-making” (Schoenfeld, 1992, p. 337). Sadly, this newer vision of school mathematics—even if touted in reform documents and by notable devotees in education—is still very much the exception rather than the rule (NCTM, 2009).

Discussion of the Effects of Beliefs: Certainty

The development of an absolutist view in mathematics may have lasting negative effects on the individuals who embrace them. Specifically, investing in math as a discipline of “objective certainty” and/or “unconditional validity” has been documented in the literature as associated with the following behaviors:

- (a) Equating learning with memorizing and following procedures (Diaz-Obando et al. 2003; Dossey, Mullis, Lindquist et al., 1988; Erlwanger, 1973; Frank, 1988; Garofalo, 1989a, 1989b; NRC, 1989; Schoenfeld, 1989; Spangler, 1992),
- (b) Accepting such procedures without question (Erlwanger, 1973; Garofalo, 1989a, 1989b; Schoenfeld, 1985; 1988),
- (c) Learning by passive reception channeled through expert facilitation (Frank, 1988; Garofalo, 1989a, 1989b; Schoenfeld, 1988), and
- (d) Believing that mathematics problems have a single, correct answer (Frank, 1988; Spangler, 1992).

Behavior (a) has been discussed at length in this paper while (b)-(d) are the main events of sections to come. Each of the behaviors (a)-(c) closely aligns with the perspective that mathematics is a finished work—one that is either neatly packaged in books or the direct result of “polished” teaching. Meanwhile, any problems that counter (d) might be labeled “ill-posed” or “extra credit.”¹⁶ Interestingly, the common thread across these learning behaviors is the association of mathematics with cut and dried objectivity. As students detach themselves from the subject of mathematics, this fuels beliefs in the discipline’s irrational, nonsensical, or even magical nature. Although absolutism has been sharply attacked as an unfit epistemology for the mathematical sciences (cf. Cobb, 1986; Ernest, 1991; 1995; 1999; Lerman, 1990), educators must grapple with the reality that *students* may see this as an outlet for success given its functionality in constrained school settings (Cobb, 1986; Schoenfeld, 1989).

Ernest (1991) presents a sharp yet eloquent critique of the absolutist view in mathematics in favor of the *fallibilist* view—fallibilism as characterizing knowledge as

¹⁶ This point is discussed further in the section *Speed: Knowledge Acquisition and Problem Completion* as an outgrowth of Frank’s (1988) distinction between *problem* and *exercise*.

uncertain and susceptible to change (also see Lerman, 1990). Although the question of certainty in mathematics surfaced at the beginning of the 20th century, three philosophical movements—logicism, formalism, and constructivism¹⁷—were clear attempts to restore mathematics as an absolutist science. In brief, Ernest rejects absolutism as a valid epistemological stance on the grounds of the failures of each of these movements. However, Ernest’s view remains widely ostracized—if not in theory then in practice—as the following comic strip from *Truth in Comics* illustrates:



Figure 1. Mathematics as universal truth

Ernest posits that relinquishing absolutism does not translate to knowledge loss. Instead, openness to fallibility permits evolution and growth in the field and opens doors to future knowledge. He cites recent developments in general relativity as evidence of the

¹⁷ *Constructivism*, as it used here, is the philosophical movement supported heavily by intuitionism and should not be confused with constructivist epistemology. For example, according to mathematical constructivists, an existence proof would require a physical mathematical entity to assert such existence. This movement failed for a multitude of reasons including inconsistencies with orthodox mathematical results as well as being founded on intuition (i.e., subjectivity).

limitless bounds one may reach by embracing knowledge as relativistic, contingent, and contextual.

Granted, several of the tenets of absolutism appear to channel the maladaptive beliefs that students hold. The net effects of these beliefs are nothing short of shocking: “In popular terms, formalism is the view that mathematics is a meaningless formal game played with marks on paper, following rules.” (Ernest, 1991, p. 10). There appears to be a trickle down effect from absolutism to subject matter portrayal—to the manner of transmission, to how the subject is conceived, to what students believe. What’s more, Ernest (1999) laments the role of textbooks’ and teachers’ push for standardization of mathematical content; this misleading objectivity diminishes the role of what he calls rhetorical or “know how” language. Although Ernest champions this language as genuine mathematical substance (beyond mere ornamentation), this again is not the widely accepted view (Ernest, 1999). As long as educators and authors continue to detach mathematics as an even-tempered and objective science, one would expect these beliefs to continue to persist and evolve in the coming years.

Speed: Knowledge Acquisition and Problem Completion

If I’m able to solve a problem in five minutes it’s OK, otherwise I give up, 5-10 minutes is the maximum.

—Lucia Mason, 2003, p. 80 (student remark)

The association of speed with mathematical experience is so common it has become *de rigueur* in educational circles.¹⁸ As crucial a role as accuracy plays in mathematics (see the previous section), the ability to generate *quick* solutions is

¹⁸ In general, this statement does not apply to graduate students of mathematics who may show great poise and determination in problem solving situations (Carlson, 1999).

indicative of content mastery in the eyes of many students. Additionally, the speed factor appears to straddle across two areas—first, in comprehension and/or learning new mathematics (one must “get it” the first time), and second, in the physical act of doing mathematics (if one knows the math it won’t take long to solve problems). Schommer (1990) organizes the first of these areas along a continuum with “quick, all-or-nothing learning” on one side to gradual/steady acquisition on the other. With respect to individuals as doers of mathematics, one may conceptualize quickness and perseverance as endpoints of a spectrum. The aim of this section is to document why gradual learning and perseverance may not be the popular views of mathematics students. Since many of the studies discussed here were cited previously, this section will be brief.

The belief that answers to math questions should come quickly is documented in some of the studies previously discussed (Frank, 1988; Garofalo, 1989b; Schoenfeld, 1988, 1989). Baroody and Ginsburg (1990) further argue that this belief, coupled with the widespread conceptions that mathematics is detached from everyday life and that only brilliant individuals can understand mathematics, plant the seeds for a future of frustration and inflexibility. As Schoenfeld (1985, 1988) admits, this view on *computational* problems poses little harm given students’ widespread exposure to routine exercises. However, a catastrophe arises when students generalize this view to *all* of mathematics—embracing the view that *all* math problems should be completed in five minutes or less (Schoenfeld, 1985, 1988). It is promising that Mason’s (2003) work shows change on the horizon. In a study of 599 high school students in Italy, beliefs were found to be eclectically mixed unlike the highly indigent views of American students from decades past. Even so, this juxtaposition should be interpreted with

caution given the widespread differences in several variables including culture, curriculum, and technology.

Together with this belief is the prevalence that smart individuals are rapid problem solvers (Spangler, 1992). As Schoenfeld (1988) found from his work with secondary and college students, if one knows the material then one will be able to solve problems in a matter of minutes.¹⁹ On a survey given to 221 high school students, the following question was asked: “If you understand the material, how long should it take to answer a typical homework problem?” The mean response was 2.2 minutes (Schoenfeld, 1988, p. 160). A decade and a half later, Mason’s (2003) large international study indicates that a chasm exists with respect to students’ expectations of speed; this ranges from believing that problems should be completed quickly to believing in persisting as long as it may take to be successful. Naturally, the convergence of the beliefs that (a) mathematics is a domain characteristic of “quick” solutions and (b) smart individuals do math quickly, presents a serious obstacle for those who are not so speedily inclined.

A fitting discussion to include here is Frank’s (1988) distinction between *problems* and *exercises*.²⁰ She views *problems* as those mathematical tasks in which courses of action are not clear while *exercises* evoke well-known procedures to deploy. She argues that the overabundance of exercise-driven mathematics has forged in students exactly these debilitating beliefs. For example, a student who has been overexposed to this system may react in one of the following ways when given a genuine mathematical *problem*: (a) giving up too soon (believing that problems should be done quickly), (b)

¹⁹ As the previous section discusses, the widespread view of *knowing* mathematics is that of recalling facts and following procedures. Thus, the speed factor resonates with this view.

²⁰ Schoenfeld (1985) went so far as to categorize the problem/exercise distinction as one of several dichotomies in mathematics education.

characterizing the task as unfair or extra credit (an oversimplified view on mathematics), or (c) approaching the problem in a mechanical fashion as if it were an exercise (Frank, 1988). It is clear that none of these approaches are particularly conducive to meaningful learning. Instead, each of them typifies a misguided view on mathematics and the activities that characterize the discipline as a whole.

Summary: Speed of Knowledge Acquisition

From this section, it is clear that many students believe that acquiring new math knowledge and “solving problems” ought to be rapid processes, perhaps free from the arduous task of thinking. It is interesting to ponder why this might be so. First, given the broad context of societal norms and expectations, the discipline of mathematics is commonly associated with puzzles, brain-teasers, or other games related to intelligence (Dunn, 1980; Gardner, 1994). At the same time, words such as *investigate*, *reason*, *conjecture*, *create* and *generalize* personify mathematical activity (Polya, 1954). These viewpoints fuse to shape a belief that if a person were to “solve” a problem in record time, he/she must be “intelligent”—perhaps brilliant. Unfortunately, Polya’s views are rarely experienced by students, who instead, are bombarded with rote exercises designed to assess mastery of small chunks of information (Schoenfeld, 1985, 1988). For these reasons, it is sensible to conclude that these beliefs develop from such experiences.

Discussion of the Effects of Beliefs: Speed

Sadly, students who embrace these views about speed may surrender all efforts after a short time (Schoenfeld, 1988). Furthermore, a teacher’s “smoothing out” of curricula—eliminating unfamiliar or problematic situations (Doyle, 1988)—only

reinforces this view by encouraging the search for artificial matches between problems and procedures (Garofalo, 1989a, 1989b). These very ideas, coupled with the notion that humans strive to reduce ambiguity in complex situations (Doyle, 1988), slowly carves out one's view of mathematics as a collection of routine tasks that should be completed with precision and speed. Moreover, if only specific methods are sanctioned by authority (Frank, 1988; Schoenfeld, 1992; Spangler, 1992), this may fail to align with students' informal (and often insightful) ways of doing math (cf. Baroody & Ginsburg, 1990; Harel & Behr, 1991). Being forced into sterile and regimented routines (and to be done unto repeatedly) can prompt a student to question whether mathematics should make any sense at all. If arbitrary methods are continuously cast upon students, this may be all the motivation (s)he needs to prematurely decide that memorization is the central survival tactic.

It goes without saying that educators' greatest concerns are the damaging consequences of holding such tacit views. If a student cannot comprehend or complete math problems within the time implicitly deemed acceptable, this may cement a false sense of inadequacy within him/herself or with the problem(s) posed (Frank, 1988). The implications of either can be detrimental and virtually irrevocable. First, the view of diminished self worth can have lasting effects on efficacy, attribution, and motivation (Bandura, 1997; Bendixen, 2002; Kloosterman & Cougan, 1994; Kloosterman, Raymond, & Emenaker, 1996; Kloosterman & Stage, 1992; Weiner, 1986). Second, the view of "quick" mathematics promotes an oversimplified representation of the nature of mathematics, particularly in the areas of structure and certainty. From this stance, beliefs concerning the speed of knowledge acquisition and performance do not stand in isolation.

Instead, they influence (and are influenced by) the concomitant beliefs discussed elsewhere in this paper.

*Student and Teacher Roles:
Hidden Messages of Classroom Mathematics*

What students mostly do is listen, watch, and mimic things that the teacher and textbook tell them and show them. If students' epistemologies are influenced at all by the experiences they have, then most students probably learn that mathematical knowledge is a form of received knowledge . . . Another probable outcome for many students is a belief that they were endowed with a low level of mathematical ability and that there is little or nothing they can do to become mathematically able.

—James Greeno, 1991, p. 81

This section's aim is to reveal the myriad beliefs that develop from years of classroom conditioning by way of student-teacher interactions.²¹ Naturally, some of the beliefs discussed in earlier sections fit into this complex web of beliefs. Greeno's words will serve as a springboard for three ideas that will be discussed in this section. First, he alludes to the absence of students' willingness to evaluate claims by way of parroting and mimicking what is learned. When assessing the validity of an idea, does one check with the teacher or convince the self? Hofer and Pintrich (1997) classify questions of this nature under the purview *justification of knowledge*. As will be illustrated here, it is widely believed that yielding to authority is the proper course of action.

Second, Greeno makes reference to students' expectation for passive reception of content. This may follow in the footsteps of rigid views of certainty and structure in mathematics. That is, if mathematical truths rest on the pages of textbooks and in the minds of educators, then its transmission may as well be as direct as possible (Frank,

²¹ Thus far, there seems to be an unconfirmed hypothesis that students develop specific views of mathematics implicitly. Even Greeno's use of language in the above quotation suggests obvious reservations with such claims. This area is revisited and discussed at length later in this paper.

1988; Garofalo, 1989a, 1989b; Schoenfeld, 1988). Embracing this view has implications for how one believes learning should take place and is commonly associated with the diminished role of sense-making. Schommer (1990) fits such concerns under the label *origin of knowledge* while Hofer and Pintrich (1997) refer to this as *sources of knowledge*. What is notable here is the extension of such a belief; that is, if math is “out there” then there emerges the concomitant belief that mathematics is unrelated to daily tasks (Baroody & Ginsburg, 1990; Verschaffel & DeCorte, 1997). This will also be treated in this section.

Finally, Greeno’s words suggest the need to reflect on the self as learner and doer of mathematics. Is mathematical ability innate or are there gradual and controlled changes that lead to lifelong improvement? Schommer (1990) uses the term *control* to refer to these questions. With respect to control, epistemological issues are sometimes blurred with source/origin and expectancy of speed while others treat these areas as independently operating dimensions (e.g., Schommer, 1994; Schommer-Aikins, 2002). Disputes aside, each of the areas addressed in Greeno’s quote intersect along the common axes of (a) knowledge acquisition and (b) teachers’ potential to mold student beliefs.

As previously discussed in the section *Certainty of Mathematical Knowledge*, students’ ways of justifying knowledge appear mostly in the form of expert ratification (Lampert, 1990); if the teacher says something is correct, then it is so. It appears that this belief is merely an amplification of related beliefs as connected to mathematical structure and its unfailing certainty. Yielding to an authority figure as a means of justification is a convenient way to couch the belief that math problems have unique answers (Frank, 1988; Spangler, 1992) obtained by the application of specific algorithms (Schoenfeld,

1992). That is, since experts know such “closed and definite” information, it is a standard act that a student check for correctness and the teacher respond committally. Teachers provide the most direct means for justification and validation; in fact, the classic didactic model of “listening attentively” claims this is so (Krantz, 1999).

In sum, students’ acceptance, willingness, and predilection to “receive” mathematics in classroom settings is well-documented in studies in which alternative didactic models draw serious resistance.²² Masked in this view might be that students fail to see thinking as an integral part of mathematics and thus, wish to be freed from this task (Frank, 1988; NRC, 1989; Spangler, 1992). Finally, viewing math as a static finished product, the transmission model of “telling” seems most fitting for knowledge acquisition. Given these perspectives, students likely internalize additional views predicated on such faulty conceptions. These are discussed next.

To begin, one may conceive that mathematical knowledge is developed by those who are gifted and creative (Erlwanger, 1973; Garofalo, 1989a; Schoenfeld, 1985, 1988, 1992). If only intelligent individuals can grasp mathematics this may engender a sense of detachment with the subject—most notably, believing that mathematics is a foreign endeavor relating little to everyday situations (Baroody & Ginsburg, 1990). This separation is supported by several studies confirming students’ inability to use contextual information or nonmathematical constraints in problem-solving situations. That is, the “mathematics” may give an answer, but one that is ill-suited to the context in which it appears. These studies are discussed presently.

²² For example, the Moore method for teaching mathematics has received both praise and criticism (cf. Krantz, 1999) while much has been written about the good and bad of calculus reform (Krantz, 1999) as well as inquiry in differential equations (Kwon, Rasmussen, & Allen, 2005; Rasmussen, 2001; Rasmussen & King, 2000; Rasmussen & Kwon, 2007).

Greer et al. (2002) provide a summative glance at French and German studies from the eighties that posed ludicrous problems to students only to receive equally ludicrous answers (e.g., “There are 26 sheep and 10 goats on a ship. How old is the captain?”). Naturally, this appears to be an outgrowth of the “keyword” strategy of adjoining mathematical operations to numbers provided in statements of the problems (Lester & Garofalo, 1982; Schoenfeld, 1992; Tsamir & Tirosh, 2002). Additionally, both Schoenfeld (1988) and Verschaffel and DeCorte (1997) summarize evidence from America’s *National Assessment of Educational Progress* (Carpenter et al., 1983) in which 45,000 students were asked the following problem: “An army bus holds 36 soldiers. If 1,128 soldiers are being bused to their training site, how many buses are needed?” It is reported here that while 70% of the students were able to do the “mathematics” (answer: 31 remainder 12), only 23% of the students reported the correct answer. Others answered with the literal response “31 remainder 12” or failed to interpret the answer by providing “31 buses” as a response (Schoenfeld, 1988, p. 150). These examples demonstrate students’ inattention to context when immersed in tasks of mathematical reasoning.

The suspension of realistic information in mathematical problem solving is so widespread that it is the impetus (and theoretical footing) for the work of Verschaffel and DeCorte (1997). Fitting here is their report of the conceptions held by fifth- and sixth-grade students *prior* to any research intervention. Through the use of a questionnaire, the authors provide ample evidence of young boys’ tendencies to ignore realistic information when solving mathematics word problems. The questionnaire consisted of 10 “problematic” items and five straightforward items (what Doyle (1988) might call novel

and familiar work, respectively).²³ The “problematic” questions were tasks which posed significant modeling and contextual challenges while the straightforward items could be solved by simple mathematical operations. Furthermore, the 10 problematic items were divided into five pairs in which one in the pair contained modeling and contextual challenges familiar to the student while the accompanying problem had a similar modeling challenge but embedded in an unfamiliar context. The results showed a mere 15% of responses across three classes (54 boys aged 10-12 years) in which realistic assumptions were adequately considered.²⁴ In short, given the constrained setting of word problems, these students suspended the use of knowledge from outside the domain of mathematics. To add, Mason (2003) reports on a similar finding. Some of the students convey the importance of mathematics in practical situations (cooking, shopping, etc.) while others believe it is nothing more than “mental exercise” to “avoid your brain getting lazy” (Mason, 2003, p. 81-82).

Finally, studies concerning the beliefs of perceived mathematical ability report findings that are somewhat inconclusive. On the one hand (analogous to Greeno’s powerful words) Schoenfeld (1989) has reported that mathematical ability is widely viewed as innate and only alterable to a minimal extent. For example, the questionnaire item *Some people are good at math and some just aren’t* had a mean response of 1.66 (n = 230) where 1 = very true, 2 = sort of true, 3 = not very true, and 4 = not at all true. This is an alarming result even if it sits twenty years removed. Regardless, beliefs persist with

²³ Aside from referencing earlier studies as original sources of many of the questionnaire items, no information on the validity or reliability of the instrument is provided.

²⁴ Sample item: “Sven’s best time to swim the 50 m breaststroke is 54 seconds. How long will it take him to swim the 200 m breaststroke?” (Verschaffel & De Corte, 1997, p. 584). Obviously, this cannot be interpreted as the multiplication problem 54 times 4. In the truest sense, issues such as fatigue, motivation and stakes (among others) should be considered.

such unwavering stability that this misguided view may even reflect reality today (Fischbein, 1987; Lester, 2002; Mason, 2004).

More recent findings render a slightly different portrait. For example, Kloosterman et al. (1996), in an analysis of elementary school children's developmental beliefs, found that most students believe that effort is an instrumental factor to success. Additionally, Kloosterman and Cougan (1994) found that all fifth and sixth graders interviewed ($n = 18$) believe that anyone can succeed in mathematics (even if the authors admit that this could be a reaction to teachers/parents *directly* feeding this information to the students!). Nevertheless, both of these findings are notable given the *National Research Council's* prevalent findings to the likes of not being "born" to do mathematics (NRC, 1989). Additionally, Mason (2003) found support for both sides; while some students believe mathematical aptitude is fixed, others feel that applying knowledge and working hard can improve ability. In short, a juxtaposition of the work of the 80s alongside newer findings suggests that students and investigators alike may be confused as to what *should* be believed (Lester, 2002; Mason, 2004)! Clearly, the research community has much unfinished business in this area.

Summary: Hidden Messages of Classroom Mathematics

The literature suggests that beliefs do not develop in isolation. Instead, they are the products of years of reflection in the classroom, with strong ties to instructional methods and peer interaction. The beliefs discussed in this section generally indicate students' preference for high comfort levels cast in routine tasks and frequent teacher feedback. Justification is primarily through validation with authority while the sources of knowledge are generally textbooks and content experts. Additionally, many individuals

determine early in life whether they are “math people.” Although recent contributions to the field show signs of alternative views, today’s mathematics classrooms have eerie similarities to the classrooms of old. Students desire clear step-by-step guidelines from the teacher to minimize thinking and if one finds success in this environment, it is determined that mastery has been achieved. Clearly, there appears little one can do to remedy the situation other than making significant changes to assessment and instruction.

Discussion of the Effects of Beliefs: Hidden Messages

The collection of beliefs discussed in this section may inculcate further beliefs about mathematics or foster in students specific modes of action. In particular, accepting passive reception as a mode of instruction may instill views about math ownership and influence how one practices mathematics. Also, with authority playing such a central role, students may see satisfying the teacher as ranking above mastery of content. In doing so, students may come to see certain mathematical acts as acceptable and others as forbidden. A few of the research findings related to these areas will be explored in this section.

First, if little input is expected from students, doing mathematics translates into reproducing someone else’s work (Garofalo, 1989a). A notable illustration can be found in Schoenfeld (1988) in which students received precisely this message while doing constructions in geometry. As students forget the steps for a geometric construction, the teacher prompts to the effect of, “What is the first thing we do? Then what? And next?” Aside from cementing the view that mathematics is rehearsal, students may conclude that creativity and originality play no part in mathematics. Someone else has discovered the constructions and it is now the student’s job to repeat them back to the teacher. It could

be argued that this disconnect with contemporary mathematics feeds into students' suspension of common sense knowledge while solving problems. In sum, students may sense that any personal insight whatsoever relates little to mathematics.

Second, doing mathematics may be equated with obtaining the answers that authorities seek. As previously discussed, Erlwanger's (1973) Benny was often out on "wild goose chases" to ensure progress through his program; Diaz-Obando et al. (2003) found that one of two students interviewed stressed the importance of pleasing the teacher; Schoenfeld (1983) speaks of students' awareness of "academic" mathematics as a forced game in which realizations such as "the teacher demand[ing] it" are reason enough to take action (Schoenfeld, 1983, p. 338). Further, Garofalo's (1989b) discussion of a class attempting the problem $3 - 2\frac{1}{2}$ reveals students' efforts to appease an uncomfortable situation (i.e., faulty suggestions) seemingly above that of actually solving the problem. Finally, Doyle (1988) comments on students' willingness to openly negotiate with teachers the expectations of classroom tasks in order to reduce ambiguity.²⁵ All of this research supports the fact that classroom experiences may be shaped a great deal by students' attempts to please and reflexively, authority's willingness to participate.

Perhaps the most interesting outgrowth of the student-teacher relationship is that some students prematurely judge the legitimacy of behavior on the grounds of mathematical structure. Most prominent here is the belief that applying a shortcut or using prior information is illegitimate—"cheating" in the eyes of some. The work of Baroody, Ginsburg, and Waxman (1983) illustrates this point well. The primary purpose

²⁵ Of course this may be interpreted as serving a dual role—that of pleasing authority and ensuring individual success.

of their work was to reveal the extent of children's usage (or lack of usage) of mathematical regularities and patterns when engaged in tasks of addition and subtraction. The three principles under investigation were (a) commutativity, (b) subtraction as complementary to addition (i.e., $3 + 5 = 8$ so that $8 - 5 = 3$), and (c) the $N + 1$ progression principle (i.e., $4 + 1 = 5$, $4 + 2 = 6$, $4 + 3 = ?$). The tasks were designed in such a way that they were fun (a game setting was used) and there were ample opportunities to use such principles. For example, the problem $7 + 9$ immediately followed $9 + 7$ and children had access to the preceding result.

Using the assumption that knowledge of mathematical structure can lead to efficient problem solving strategies (Wertheimer, 1945), Baroody et al. (1983) report some interesting findings but none more shocking than students' personal beliefs: the investigators found that several children failed to apply principles of structure not because of lack of knowledge but for *fear of being caught*. What might be inferred here is that mathematics is more about carrying out specific processes rather than pattern recognition—even in the elementary years. Recognizing such regularities minus the labor of counting/computing is viewed by some as “cheating.”

Cobb (1985) found analogous results while investigating two first graders' responses to mathematical tasks given over a two-year period. One of the two students refrained from using previously exposed principles for fear it was dishonest. In other words, what might be viewed as sophisticated or productive by the teacher was perceived as potentially “naughty” to the student. If such incongruities remain hidden in the world of classroom mathematics, students may build an extensive catalogue of misbeliefs deeply rooted in early exposure to mathematical primitives.

From these studies, it appears that the reluctance to incorporate previously learned mathematics may be an artifact of elementary level mathematics (since beliefs are still developing and content is dominated by computation). An area especially thin on research is whether this view continues into later years including secondary and post-secondary levels. However, one could argue that this belief morphs into one concerning the importance of *mathematical form*. In Schoenfeld's (1985, 1988) work, there is much discussion of secondary students engaging in *informal* proof with great success. However, students are often heard uttering "Now let's do it the right way" indicating the importance of form and convention when writing proofs. The similarity to the studies here is undeniable. A student who uses the knowledge $5 + 5 = 10$ to determine $5 + 6 = 11$ might feel empowered by this realization but simultaneously believe the "right way" is via a systematic counting technique.

The Bottom Line: What Students Learn from the Mathematical Experience

The goal of this section is to articulate, in the broadest terms, what students take from their classroom experiences with mathematics. While the body of this paper addresses the beliefs held by a diversity of groups, two focal points will be discussed here as germinating from these experiences. First, it appears that epistemological beliefs *encourage* particular outlooks on mathematics; to argue that these beliefs *promote* or *cause* such outlooks might be too strong a claim. The literature reports that students embrace these views as they manage the daily realities of academic mathematics. Under certain conditions, these beliefs act as concealed coping mechanisms as students redirect their efforts to learn mathematics to meet social needs. Second, beliefs founded on misguided principles can have devastating and permanent effects. For example, few

would deny the American public's disdain for mathematics and its applications.

Arguably, even a brief exposure to such negativity can infiltrate students' thinking and spawn a lifelong association of mathematics with the notion of having the "capacity" to engage in it. Perhaps no discipline in the traditional liberal arts education bears this burden as much as the subject of mathematics.

Functioning in the classroom setting

Students pick up the rhetoric for how to deal with mathematics in the school and they have developed mechanisms for being successful in this setting (Cobb, 1986; Garofalo, 1989a; Schoenfeld, 1983). These mechanisms are precisely the beliefs that comprise the heart of this review. Cobb (1986) argues that the daily microcosm of mathematics classrooms fosters transformations in students to the effect of ritualistic acts over time. This may instill in students the need to reconcile problems that are chiefly social—not mathematical—in nature (e.g., appealing to authority, appearing intelligent in the face of others, playing the "academic game" of mathematics, etc.). One could argue that these social factors systematically map to student beliefs (e.g., math is best learned passively, math is created by the gifted, one must get the work done and please the teacher, etc.). Cobb's words amplify this very idea:

The child's overall goal might then become to satisfy the demands of the authority rather than to learn academic mathematics per se. This goal can be achieved, at least in the short term, by either covertly constructing and using self-generated methods or by attempting to memorize superficial aspects of formal, codified procedures. If the latter approach is adopted, mathematics becomes an activity in which one applies superficial, instrumental rules. (Cobb, 1986, p. 7)

While in the classroom setting, multiple factors define what is problematic for the learner. For example, there are (a) situational factors of classroom mathematics (e.g., sociomathematical norms, teacher demands), (b) circumstances which the learner creates

for him/herself (e.g., what is the learner trying to accomplish?), and (c) epistemological beliefs about mathematical knowledge and knowing.

Cobb further acknowledges that learners of mathematics, under the continuous conditioning of the system, adapt to the constraints put forth and find ways to cope with almost any difficulty. Students formulate what one might call meta-problems—problems that appear mathematical in nature but lack intellectual foundations in lieu of economic or social gains.²⁶ Given this ideology, students set personal goals and perform the necessary activities to achieve these goals. This complex internal struggle provides a context from which one’s beliefs can be inferred (Cobb, 1986). Of course, this entire discussion begs the question: *If the constraints of the classroom were somehow lifted (or radically altered), would these beliefs continue to persist?*

Is math for everyone?

A majority of the beliefs cited align with students’ implicit association of “mathematical satisfaction” with *competence*—irrespective of the emotions that accompany it (e.g., exhilaration, frustration, or something in between). The work of Stodolsky et al. (1991) offers a perspective on how students interpret pleasant/unpleasant experiences in math/social studies in different ways. For example, it is reported that the reason for enjoying math is *ease*; for social studies, it is *interest*

($\chi^2 = 31.261$, $df = 4$, $p < 0.001$) (p. 103). On the other hand, reasons for disliking math were overwhelmingly due to *difficulty*; for social studies it is *boredom*

($\chi^2 = 22.326$, $df = 5$, $p < 0.001$) (p. 103). In sum, mathematics as a subject has an

²⁶ Current work by Harel (2008, in press) emphasizes the importance of intellectual necessity in promoting deep, meaningful learning.

undeniable connection with the human capacity to do it: “The only way that students can interpret math is in terms of their ability to do it, whether as a function of their performance (successful/unsuccessful) or of the task (easy/hard)” (Stodolsky et al., 1991, p. 112). Some of the beliefs discussed in this paper (e.g., only smart individuals can do math, mathematics aptitude is innate, etc.) resonate with this view that mathematics is restricted to the intellectually privileged.

There are numerous beliefs that seemingly connect to this view depicted above. As discussed earlier, a learner may come to believe there is something “wrong” on a personal level if math problems cannot be solved with the greatest of swiftness (Schoenfeld, 1988). Further, the dominant theme of being able to “do” mathematics alongside the view of being “endowed” with this gift may further explain why students make few attempts to understand mathematics. Even today’s students are generally not expected to make sense of mathematics as they engage in it; this is evidenced by discussions of conceptual and procedural understanding that continue to spark heated debate (cf. Baroody, Feil, & Johnson, 2007; Star, 2005). Mathematics, from a student point of view, is chiefly an activity in which *completing the work* and/or *pleasing the teacher* become central motivators to success (Doyle, 1988).

Finally, as a direct consequence of having math “passed down” from authority, students may come to the conclusion that they simply cannot learn the subject of mathematics without outside assistance. When Stodolsky et al. (1991) asked the question *Can you learn the subject on your own?* only seven of 60 students replied “yes” for mathematics whereas 24 of 60 replied “yes” for social studies. It seems that the view shared by many is that mathematics is something beyond the grasp of average-ability

individuals. From this perspective, the oft heard expressions “I was never a math person” or “math is for geniuses” become more than just beliefs—they emerge as convenient outlets to mathematical evasion. In time, these expressions shape the views of others and grow to be accepted as the norms we see and hear today.

AN IMPLICIT ASSUMPTION?

The research findings on students’ epistemologies offer an interesting portrait of how students view, think about, and act in mathematical situations. Presently, the field is well regarded as evidenced by the steady stream of new and significant contributions (Maasz & Schlöglmann, 2009; Sriraman, 2008). Through the years, educational researchers and mathematicians have documented the presence of mostly ill-suited beliefs and the subsequent impact on students’ thinking and actions. Despite this, a notable *assumption* has been embraced by the majority of thinkers from this period. This assumption is stated below.

Assumption. Students’ epistemological beliefs are acquired and/or honed through classroom experience(s) and/or teaching.

What is especially interesting about this assumption is that many researchers rely on the very merits of this hypothesis to infer this directionality. Without question, experts unanimously conclude that these *unspoken* or *unintended* messages arise from teaching and classroom conditioning (Diaz-Obando et al., 2003; Doyle, 1988; Garofalo, 1989a; Greeno, 1991; Kloosterman et al., 1996; Schoenfeld, 1988, 1989; Spangler, 1992). Detractors can be found but are slim by comparison (Franke & Carey, 1997; Ruthven & Coe, 1994).

Although this assumption appears justified given the consistent findings on students' beliefs, the anomaly remains as to why some students are unmoved by teaching that is designed to promote an alternative view. That is, the naïve ideologies cited throughout this paper seem to emerge irrespective of teaching. This discrepancy urges researchers to investigate the *origin* of such views, assuming that these views could arise from something other than teaching. This proposal in no way aims to discount the research that has shaped our understanding of students' beliefs in mathematics. Rather, it intends to explore whether such an assumption can or should be embraced since it appears to have little empirical support. With this realization in mind, we turn to the conclusion of this paper but through the eyes of researchers who embrace this assumption.

CONCLUSION

Overall, the articles and reports cited in this paper have fared well in addressing the question, "How do students' epistemological beliefs about mathematics shape their ways of thinking about and doing mathematics?" Emanating from these "ways of thinking" and "ways of doing," students likely leave the formal study of mathematics with a lasting impression of what the discipline is all about. These impressions specifically point to the question "In what ways do these beliefs have an effect on what is ultimately learned?" Given the organization of this paper, it is stylistically convenient to revisit these questions in the context of the initial framework proposed.

With regard to mathematical structure, a large portion of the research highlights students' views of mathematics as a disconnected body of information governed by rules and procedures. Consequently, students' ways of thinking are dominated by mapping

“new” problem situations to ones previously encountered—usually in search of an artificial congruence. As a result, “doing” mathematics translates to applying previously learned algorithms. Unfortunately, this belief and its resulting effect on students’ thinking sends a clear message about learning: *One had better memorize these rules and procedures if one plans to be successful.* The damage comes full circle when students see mathematics as detached from any sort of thinking activity—a paradox to say the least.

Additional research on mathematical structure conveys students’ artificial separation of theory/proof from exploration/discovery. The research documents the widely accepted view of “proving” as merely validating pre-established truths. That is, proving becomes a ritualistic act with little room for invention. This is a sobering finding given that proof is a place where professional mathematicians invent, derive, and sometimes extract meaning (e.g., geometric insight) into mathematical statements. For students, experiences with proof suggest that mathematics is a game and that specific rules are to be followed. Even worse, students may view such activities as ones only played in mathematics classrooms but not in daily life.

Student beliefs about the stability or certainty of mathematics present yet another area in which lasting negativity pervades. Much like the beliefs about the structure of mathematical content, beliefs that mathematics is indisputable and unchanging encourage learners to memorize preset procedures for later retrieval. In effect, “doing” math is synonymous with regurgitating information once provided. The most shocking outgrowths of such beliefs are what students ultimately learn from these experiences. First, believing math is static by nature fosters a willingness to accept procedures without question, again jettisoning “thinking” from mathematical activity. Second, learners may

feel that passive reception is the optimal way to learn the subject given its status as a finished body of work. Viewed from this angle, “teaching as telling” seems especially fitting.

Speed in mathematics appears to be an important part of how students (a) view the subject and (b) perceive their likelihood of success (even if newer findings show some degree of improvement). When speed is considered important in mathematical learning, thinking is discounted in lieu of efficiency. One can see close ties to some of the previously discussed beliefs since speed and quickness are associated with “knowing mathematics cold” through routinization and memorization. Clearly, such a view has implications for how one organizes his/her understanding of the subject (structure). As previously discussed, doing mathematics quickly is consonant with finding the algorithm/procedure that is suitable to the task on hand. What is ultimately learned from these experiences is a triad of perhaps the most devastating of beliefs:

- (a) fast problem solvers are “smart” individuals or just all-around “math” people,
- (b) *all* mathematical tasks should be solved quickly (a false generalization), and
- (c) individuals should give up or ask for help if a solution does not come quickly.

One might argue that these perspectives manifest themselves at the macro level as “innumeracies” (Paulos, 1988).

Despite the implicit assumption previously mentioned, much research has added to our understanding of the *potential* effects of classroom dynamics and teacher actions as reflected in student behaviors. These “hidden messages” (as they are often called) center on students’ unwillingness to evaluate mathematics for themselves by way of parroting back what the teacher has taught. Mathematical activity is then largely a byproduct of seeking and displaying outcomes that authority figures deem acceptable.

The bottom line is that learners not only discover the rules of classroom mathematics (including habitually yielding to authority) but come to understand mathematics as an act in which structure, speed, and exactness are the cornerstones to success.

THE ROAD AHEAD: THE CHANGING FACE OF BELIEFS RESEARCH

As work initiated for this paper, the writer was familiar with certain facets of the beliefs research in mathematics education. Specifically, the work of Alan Schoenfeld is well regarded in educational circles as is the surge of accompanying reports from the late 80s and early 90s related to students' "shocking" beliefs about mathematics. Hence, this particular research served as a starting point from which one could backpedal to earlier work (Baroody et al., 1983; Erlwanger, 1973; Lester & Garofalo, 1982). Additionally, the chapters by Lesh and Zawojewski (2007) and Philipp (2007) from the recently published *Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007) as well as the current volumes by Maasz and Schlöglmann (2009) and Sriraman (2008)—both with strong emphases on beliefs—contain fresher perspectives but firmly situated in the research from decades past.

Upon realization of the *Second Handbook's* failure to address students' beliefs directly, one must wonder what this could mean.²⁷ Have student beliefs fallen off the radar of researchers' interest? Is the field considered "finished" or so saturated that there is little more to contribute? Or could it be that educational researchers now respect the field of beliefs enough to include it as a component within their area(s) of specialization? It appears, at least through the eyes of several experts, that the answers to these questions are a tentative "yes." It is for these reasons that the reader will find an abundance of

²⁷ There is a chapter whose focus is teacher beliefs.

work from the 80s and 90s as this research paved the way for establishing “student beliefs” as a legitimate area of study in mathematics education. This “classic” research has been influential in shaping the field and it is likely to remain the authoritative source for theoretical and empirical work on student beliefs for years to come.

Informal discussions with Peter Kloosterman (personal communication, 2008) and Frank Lester (personal email communication, 2007) reveal that “student beliefs” remains an extremely important research strand in mathematics education. However, both Kloosterman and Lester contend that the foundations for much of this research is already in place (i.e., the research reviewed for this paper) and that newer studies are (a) theoretically grounded in such work and (b) show signs of a more concentrated focus. It is not that researchers no longer take interest in student beliefs. In fact, general research on beliefs similar to that from the 80s and 90s is still common (cf. Diaz-Obando et al., 2003; Francisco, 2005; Liu, 2009; Schommer-Aikins, Duell, & Hutter, 2005) but the results of this work seem to add little to the community’s understanding aside from replicating the findings of the past.

In contrast, there is noticeable momentum in beliefs research in conjunction with other research strands in mathematics education (e.g., students’ understanding of calculus, instructional devices and/or strategies, periphery affective constructs, etc.). Examples of such multiplicity include studies of beliefs about proof (Furinghetti & Morselli, 2009), integral calculus (Rösken & Rolka, 2008), real numbers and infinitesimals (Ely, 2010), learning tools/devices (Nabb, in press), student participation (Jansen, 2008), equity and social justice (Povey, 2002), et cetera. These finer-grained studies carry the assumption that students may hold the beliefs as detailed in this paper

but aim to reveal more targeted beliefs as situated in the context in which they emerge. From here one may try to forge a connection to the broader findings from years past in an effort to understand why students view mathematics the way they do.

Personally, it is felt that we may not find today's student as seeing "mathematics as memorization" as documented years ago. Instead, we may find specific areas of mathematics infiltrated by these maladaptive views while other areas are less affected. This does not contradict the research of old but it refines this research by sharpening the research questions and capturing beliefs in context. Time will tell if arranging the pieces to such a complex puzzle will generate connections to earlier beliefs research in mathematics education.

Aside from more refined studies, the field of beliefs has splintered into various subdivisions. Two that are receiving much attention at this time are *teacher beliefs* (both pre-service and in-service) and *belief change* (for both students and teachers). Both of these areas, in some sense, indicate the passing of "student beliefs" in the eyes of some and the beginning of several new chapters of work. Such movements also convey the degree of respect that the field of student beliefs has earned (given that a great deal of this new work looks to the old for stimulation and affirmation). Specifically, recent work in belief change shows traces of the "unhealthy" beliefs depicted in this paper and, most important, indicate that educators are interested in actually doing something about it. The next decade and beyond will extend the boundaries of our pursuit of knowledge, adding to our understanding of what students and teachers believe and perhaps why they believe it.

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