

Question 3

Computer algebra systems (CAS) have the potential to reorganize, reshape, and resequence the mathematics that is learned at the undergraduate level. To realize this goal, questions that address the core principles of usage, point of entry, and curriculum should be asked: How should CAS be used in undergraduate mathematics? Is there an appropriate time to introduce CAS into the learning experience? Finally, to what degree does CAS influence mathematical content and the manner in which this content is taught?

Keith A. Nabb

Illinois Institute of Technology

Mathematics & Science Education

February 2010

TABLE OF CONTENTS

INTRODUCTION	1
LITERATURE SELECTION AND ORGANIZATION	2
THEORETICAL DEVELOPMENTS	4
Instrumental Genesis	4
Cognitive Technologies: Epistemic or Pragmatic?	6
The Technical/Conceptual Divide.....	7
Constraints, Boundaries, and Obstacles	10
CAS: UTILITY.....	11
CAS: WHEN TO IMPLEMENT?	21
CAS: CURRICULUM AND INSTRUCTION.....	31
CONCLUSION.....	48
REFERENCES	53

INTRODUCTION

The term *computer algebra system* (CAS) is generally used to describe the diverse class of technological tools equipped with numerical, graphical, and symbolic capabilities. These tools can appear as computer software programs such as *Mathematica* (Wolfram Research, 2009) and *Maple* (Waterloo Maple, 2009) or as graphing calculator interfaces such as the TI-92, Voyage 200, or TI-Nspire (Texas Instruments, 2009). By and large, they are assumed to hold great promise in enhancing mathematical teaching and learning (Blume & Heid, 2008; Fey, Cuoco, Kieran, McMullin, & Zbiek, 2003; Heid & Blume, 2008; NCTM, 2000; Zbiek & Heid, 2009). Apart from standard graphing calculators which possess only numerical and graphical functionality, it is the *symbolic* capacity of CAS and its connectivity with numerical and graphical functionalities that has captured the attention of researchers and teachers internationally.¹

Given the potentials of such sophisticated instruments, several fundamental questions have emerged concerning its presence in mathematics classrooms. One question concerns that of utility; that is, how should such devices be used in the learning of mathematics? Some espouse that the potential of CAS rests in “freeing” students from mundane drills so that increased energy may be channeled into thinking and reflecting on the mathematics learned. Those opposed caution that such a use threatens the acquisition of basic skills necessary in the learning of future mathematics. Still others take a neutral position in remaining optimistic about CAS but are troubled by the obstacles and technical challenges that teachers and students are likely to encounter.

¹ By *symbolic capacity*, researchers generally speak of the ability of CAS to manipulate algebraic expressions, test for equivalence, generate answers in exact form, and the like.

A similar tension centers on *when* it is appropriate to use such tools. As above, some claim that too early an exposure to CAS may hinder students' mastery of basic skills while others advocate its early and continued use. Finally, given a device that renders mastery of algorithms somewhat questionable, some researchers offer alternative perspectives on what we should be teaching as well as how we should be teaching it. Given the presence of powerful technology, each of these areas hints to modifications to traditional mathematics curricula as well as to a reformed pedagogy.

In light of the questions above, rarely have researchers addressed one of the areas in complete isolation—for it appears that findings in one domain can adequately inform another. As will be seen throughout this paper, how a teacher uses CAS may be a function of the curriculum in which it is embedded. Moreover, whether CAS is introduced before, after, or as a skill is developed has close ties to one's teaching philosophy and epistemological stance on mathematical knowledge. Given the natural interconnectivity of the questions posed, it is this paper's intent to accomplish two goals—first, to provide a brief sketch of some important theories, and second, to survey the literature pertinent to the topic of this paper. Drawing on both theory and empirical research, this sets the stage for a richer understanding of the problems that currently plague the field.

LITERATURE SELECTION AND ORGANIZATION

The research concerning computer algebra use in mathematics education is voluminous. It contains a variety of information sources including literature reviews, theoretical pieces, systematic research studies, and opinion papers. Although it could be argued that most research domains are compartmentalized as such, the CAS research is

especially splintered in this respect. Specifically, efforts to connect research and practice seem to lag behind in comparison to other research domains (Zbiek, 2003). Moreover, because the field is generally considered to lack cohesion with respect to unifying theories (Zbiek, Heid, Blume, & Dick, 2007), no attempt has been made to intentionally exclude research based on its classification or origin. The reader will find the literature to be of an international flavor, with heavy influences from Austrian, Australian, French, and North American authorities on CAS.

Opinion pieces—often nonempirical, subjective, and anecdotal—were not avoided due to their nature. These pieces often provided panoramic views of perspectives (at the time of publication) and are important to shaping the field. Moreover, many such contributions have been sources of heated debate and likely spurred systematic research into CAS not only with respect to teaching and learning but with emphases on software design and pedagogy (cf. Buchberger, 1989, 2002; Goldenberg, 1988). Even so, it is important to realize that this work remains on the periphery of such a literature review.

The organization of this paper is as follows. To begin, a discussion of important theoretical contributions will be provided. Since most of the empirical studies discussed in the body of the paper embrace one or several of these perspectives, it is important to provide the reader some background information on such theories. In the case where theory is absent from a research study, these viewpoints still serve as referents from which to gauge the meaningfulness of the study. However, this section will be brief in order to maintain a central focus on the research questions. Next, the paper turns to the three main concerns with respect to (a) CAS use, (b) timing of implementation, and (c)

curriculum and instruction. Amid these discussions, important findings are assembled from these works and explained through the eyes of the theories mentioned earlier. Given these new insights, a synthesis of these reports conveys a general lack of consensus across the field and the growing need to address these differences.

THEORETICAL DEVELOPMENTS

Instrumental Genesis

More than any single theoretical construct in the literature, the idea of *instrumental genesis* has served as the underpinning of research on mathematical learning in CAS environments. Emanating from the work of Vérillon and Rabardel (1995), this idea asserts that using any tool—albeit a hammer, a drill press, or a computer algebra system—is rarely spontaneous and automatic. A key factor here is the distinction between the *artifact* (the tangible manmade object) and the *instrument* (the psychological tool used in acts of learning). It is only once the user has been able to adopt the physical artifact for a meaningful purpose that this genesis begins to unfold:

...a machine or a technical system does not immediately constitute a tool for the subject. Even explicitly constructed as a tool, it is not, as such, an instrument for the subject. It becomes so when the subject has been able to appropriate it for himself. (Vérillon & Rabardel, 1995, p. 84-85).

Although Vérillon and Rabardel make no mention of computer algebra and only occasionally reference math/science education, the widespread application of their theory has cast much light on the field. It has served in explaining many of the potentials and pitfalls of adopting, implementing, and understanding the impact of technology on students' mathematical thinking.

Instrumental genesis is the result of two processes—*instrumentalization* and *instrumentation*. The evolution and progression of each process intertwine in complex ways (Artigue, 2002; Drijvers & Trouche, 2008; Guin & Trouche, 1999; Hoyles, Noss, & Kent, 2004). Instrumentalization conveys how the learner shapes the technology (i.e., the user readies the technology) while instrumentation attempts to explain how the technology shapes the user (i.e., the agent develops schemas for the appropriate use of the technology). Together, this dialectic results in the instrumental genesis, encapsulating how an artifact transforms into a *bona fide* tool.

Although a large amount of research focuses on precisely this CAS-agent microgenesis, it is interesting to note that advocates and critics alike have been forthright in expressing concerns of what are sometimes interpreted as conflicting adaptations of Piagetian and Vygotskian schools of thought (Bowers, Brandt, Stovall, & Vargas, 2009; Monaghan, 2005; Vérillon, 2000).² Despite this tension, it is generally agreed that the process of instrumental genesis is unique to the individual and evolves gradually with mathematical activity (Guin & Trouche, 1999). Those not attuned to this genesis may erroneously pigeonhole unsuccessful users of CAS as simply “unable” to use the technology. In contrast, instrumental genesis would suggest that these individuals have not yet developed the schemes for its effective use (Artigue, 2002). Despite the enormous importance of this theoretical construct, there are notable incongruities in

² As Piaget’s (1970) genetic epistemology delineates, individuals form concepts, elaborate these concepts into coherent structures, and then restructure them into powerful schemas. Here, there is a noticeable resonance with instrumentation regardless of the external existence of a tool. In contrast, a hallmark of Vygotsky’s work is the examination of the social interactions that individuals have with more knowledgeable beings as seen in culturally developed systems and psychological tools for thinking (Vygotsky, 1930/1985). Viewed from this perspective, instrumentalization conveys the user’s directive attention on the artifact resulting in this eventual transformation. Embracing the validity of both positions, Vérillon and Rabardel (1995) support the view that cognition is internal yet mediated by environmental factors.

associating it with the bulk of research conducted in the United States. This is discussed elsewhere in the paper.

Cognitive Technologies: Epistemic or Pragmatic?

The view of CAS as a computation tool whose sole purpose is to solve mathematics problems is fortunately not the widely espoused view. Researchers are generally concerned with CAS and its intrinsic value on the educational experience—specifically, its ability to play a role in students’ learning and understanding of mathematics. Pea (1987) uses the term “cognitive technology” to convey the idea that such technologies can assist the user in “learning how to learn” (Pea, 1987, p. 111). These cognitive tools can leave traces of student work, foster reflection on such work, elicit what-if scenarios, and orchestrate other formative means of thinking-in-action. Pea interprets the term *technology* broadly as any invention that has provided the means for future advancement in the civilized world (e.g., symbols, written language, theories, artifacts, and the like). His central thesis is that computer technologies can explicate internal thinking processes, and, in turn, provide the learner a tangible means of reflection. He makes quite clear that the *cognitive* potential of computers as technologies sets them apart from other technologies. For example, although a pencil may come to one’s aid in reproducing a memorized list, it does so exclusively in an organizational way. In no way does the instrument stretch mental capacity.

Having benefited from the above perspective, researchers have examined the dual affordances of CAS—specifically, (a) the efficient machine output of mathematical solutions and (2) the genuine reflections from CAS that augment the learning experience. The conceptualizations used to frame these affordances are the *pragmatic value* and

epistemic value of mathematical activities with technology (Artigue, 2002; Lagrange, 1999b; 2003; Ruthven, 2002). A technique's *pragmatic value* centers on its ease of use and efficiency in accomplishing a task while its *epistemic value* concerns its potential to enrich the user's understanding of the mathematics at hand. For example (even in the absence of technology), using a highly routine mathematical procedure may allow the learner to bypass thinking; this is indeed pragmatic but appears to have low epistemic value. Artigue (2002) argues that the use of CAS in mathematics classrooms results in an imbalance to this didactic model: "Techniques that are instrumented by computer technology are changed, and this changes both their pragmatic and epistemic values." (p. 249). Debates in this regard are often filtered through the means of the conceptual and technical aspects of the activity. Because these ideas are a mainstay in the literature, it is to these aspects of the theory that we now turn.

The Technical/Conceptual Divide

In early North American studies, there appeared a predominant theme that CAS could be used to outsource tasks of mathematical drill so that students could focus on crafting solution methods and interpreting the results (Heid, 2003). Specifically, some studies called into question the widespread view that procedural mastery needs to precede conceptual understanding (Heid, 1988; Palmiter, 1991). In light of these findings, an upheaval to traditional mathematics curricula seemed imminent.

Skepticism was widespread but two important findings were vital to easing the concerns of CAS's "intrusion" in mathematics. First, the use of CAS does not, in general, weaken students' abilities to perform routine algebraic manipulation (Ayers, Davis, Dubinsky, & Lewin, 1988; Heid, 1988; Hillel, Lee, Laborde, & Linchevski, 1992;

Palmiter, 1991). Even more important, this general finding transcends multiple grade levels as well as ability levels (cf. Heid, Blume, Hollebrands, & Piez, 2002). Second, students' conceptual growth and understanding are not lessened as a result of CAS use (Heid, 1988; Judson, 1990). In fact, O'Callaghan (1998) found learning in a CAS environment to foster deeper conceptual connections of functions in comparison to similar learning in the absence of CAS. Despite these findings, teachers' marginal use of computer algebra (Artigue, 2000) may be explained in some measure by the fact that having "no effect" or "minimal harm" is hardly a reason to change: "Unless an improvement occurs in some aspect of mathematics learning [with CAS], the argument for change is not compelling." (Heid et al., 2002, p. 587).

Newer research findings (Guin & Trouche, 1999; Lagrange, 2003) alongside reflective commentaries (Artigue, 2002; Ruthven, 2002) suggest that the perceived usefulness of CAS in supplanting the technical in favor of the conceptual may be overstated. For example, some studies have found that technology does not invoke reflective thinking on its own (Guin & Trouche, 1999; Hoyles & Noss, 1992) while others remind us that using CAS is challenging in and of itself for students and teachers (Drijvers, 2000; 2002; Lagrange, 2003). Together, this shines the spotlight on new aspects of technical skill that consume the user.³

An example from Lagrange (2003) is worth mentioning here. He discusses a fairly traditional optimization problem which can be solved graphically, numerically, or analytically:

A man wants to build a tank. The walls and base of the tank are to be made of concrete 20 cm thick, the base is to be a square, and the tank must contain 32 cubic meters. Let x be the horizontal dimension of the side of the inner square, and let h be the inner vertical of the tank,

³ The theoretical construct of instrumental genesis is most likely a byproduct of this awareness.

both measured in meters. What should be the values of x and h to use as little concrete as possible? (Lagrange, 2003, p. 279).

Given the abundance of such common textbook problems, there is clearly little gain in employing CAS on such a problem. However, a more generalized problem in which the thickness of the wall and the volume are expressed as *variables* adds a noticeably complex layer to the analysis. In particular, students cannot graph the function due to the presence of too many unknowns whereas algebra systems can easily handle the analysis and potentially spark insights connected to the more refined problem. Moreover, students may see that their solution(s) in CAS depend on some variables but are independent of others (i.e., the inner wall length is dependent on volume but independent of the wall's thickness). Such insights are surely lost in conventional analyses or fail to even warrant consideration in a non-CAS environment; the inherent algebraic complexity is primarily to blame.⁴

This example presents interesting middle ground with respect to the research cited above. On the one hand, CAS is indeed used to outsource algebraic skills in favor of the interpretive aspects of the problem, but on the other hand, this particular exercise harnesses a new kind of mathematics that carries potentially new understandings. For example, students may seek dependencies between variables instead of the value(s) of the variables per se. Lagrange argues that such generalized problems encourage a machine *technique* for solving problems with CAS—making explicit the interdependence and co-evolution of technique and concept in CAS environments. Tasks of a similar nature have intrinsically high pragmatic and epistemic value and systematic research studies point to the complementary fashion of technical and conceptual knowledge afforded by the CAS

⁴ It appears that Lagrange is claiming that there is no longer a need to restrict students' diets to problems that are feasible for hand computation. Boyce and Ecker (1995) also embrace this view.

and pencil-and-paper environment (Zbiek, 2003). Studies that focus on such complementarities (Kieran & Drijvers, 2006; Lagrange, 2003; Pierce & Stacey, 2001) are discussed in the main body of this paper.

Constraints, Boundaries, and Obstacles

The important work in the areas of instrumentation and instrumentalization has fueled researchers to explore the barriers posed by the use of CAS. Anytime a user's actions are constrained by or filtered through a learning tool, the danger of magnifying the specificity of the content learned is very real (cf. Hoyles et al., 2004). The result is an added challenge for users in mapping this situated knowledge to the broader knowledge they are trying to acquire. This is a view generally embraced by Drijvers (2000, 2002), Guin and Trouche (1999), and Hoyles et al. (2004). Given the centrality of such a concern, it further supports the importance of the teachers' role in instrumental orchestration (Guin & Trouche, 2002); that is, students may encounter additional technical difficulties with CAS and need guided assistance in moving past such barriers.

Drijvers (2000, 2002) has been a leader in identifying obstacles that students are likely to encounter in CAS environments. He defines an obstacle as "a barrier provided by the CAS that prevents the student from carrying out the utilization scheme that s/he has in mind. As a result, the obstacle stops the process of shifting between the 'pure' mathematics and the problem situation." (Drijvers, 2000, p. 195). Common obstacles revealed in his work include how one copes with (1) unexpected/ill-conceived output, (2) the seemingly arbitrary discretions of CAS, and (3) CAS's refusal to execute commands.

The above obstacles are centerpieces in discussions of the ‘black-box’ nature of CAS⁵ (Bossé & Nandakumar, 2004; Buchberger, 1989, 2002; Child, 2002; McCallum, 2003) while other obstacles might be considered more global in nature (Drijvers, 2000, 2002). In later work, Drijvers (2002) documents obstacles that arise when the technical and conceptual components of an activity clash “...either because the technique is not accompanied by appropriate meaning and conception, or because the technical skills for the performance of a conceptual idea are lacking.” (p. 224).

Following a different trajectory, Hoyles et al. (2004) theorize that students who learn in a CAS environment (or with any other technology for that matter) learn mathematical content that is *constrained* and *bound* by what the tool can and cannot do. In essence, the mathematics that is learned is woven into the fabric of the interactions and experiences with the tool. Quite naturally, concerns are raised regarding the possible incongruities of this “context-bound” mathematics as contrasted with “official” mathematics.⁶ Although this view is somewhat removed from the introspection of cognitive obstacles (Drijvers, 2000, 2002), it is interesting to note that the two perspectives coalesce in drawing distinctions between two types of mathematical understandings.

CAS: UTILITY

The unadorned words of Heid, Hollebrands, and Iseri (2002) convey a central dilemma concerning the use of CAS: “What place does this powerful technology have in

⁵ Bossé and Nandakumar (2004) discuss further how students may interpret such unexpected actions into deep misconceptions.

⁶ Goldenberg (1988) and Dick (2008) convey similar concerns but with respect to graph interpretation theory and pedagogical/mathematical/cognitive fidelity, respectively.

our classrooms?” (p. 210). Although the question appears simple on the surface, many research studies and theoretical commentaries have cast light on the issue only to reveal little common ground. In this section, five distinct uses—black box, white box, amplifier, discussion tool, and catalyst for reform—will be discussed along with the field’s main empirical findings. The aim is to bring to light these different perspectives as well as reasons why specific viewpoints might be favored in the long run. The section closes with the observation that these uses may be viewed in the form of a nested model ranging from rudimentary use with straightforward implementation to a sophisticated use whose potentials are only beginning to be understood.

Black Box

The use of CAS to produce answers to mathematical questions with little attention to reasoning has received widespread criticism in mathematics education. The term “black box” (relative to computer algebra) was introduced by Buchberger (1989) to convey precisely this use—CAS as another authority figure in the classroom generating results sans the *how* or *why*. Without knowledge of the underlying mathematics, many agree that the consequences of such use are disastrous for education and beyond (Bossé & Nandakumar, 2004; Buchberger, 2002; McCallum, 2003). With only a few keystrokes, CAS may display inconsistent, unpredictable and even erroneous results as interpreted by the user. In a very real sense, CAS hijacks the user’s input and performs mysterious operations—sometimes not intended by the user. Of course, the bright side of “black box” usage is the spark of curiosity provoked in some students (Boyce & Ecker, 1995; Heid, Hollebrands, & Iseri, 2002) but otherwise, one must be careful not to cultivate an anxiety-inducing view of a subject that is already widely conceived as

mysterious (cf. Paulos, 1988). Finally, the black box approach can serve the purpose of solving problems that literally stretch human capacities to their limits—the message being that CAS can handle exceptionally intricate problems. Regardless, such a function has relatively low epistemic value and appears to add little to the educational experience as a whole (Artigue, 2000).

*White Box*⁷

Critical of the above use of CAS, many researchers and practitioners advocate the informed pedagogical use of computer algebra—what has become known in the literature as “white box” (Buchberger, 1989; Child, 2002; McCallum, 2003). For example, Heid and Edwards (2001) discuss white box usage in the context of solving linear equations. Given an equation such as $3x - 4 = 7$, a student may add four to both sides ($3x = 11$) but then decide, somewhat prematurely, to subtract 3 from both sides. Were this suggestion made in an ordinary classroom setting, it would likely be squashed in favor of automated suggestions such as “divide by three” or “multiply by one-third.” In short, a potential learning experience may be lost. CAS, on the other hand, will execute precisely the student’s requests (Figure 1, left) so a learning experience awaits the user even amidst this suboptimal move. Figure 1 (right) illustrates what the learner might do after this realization.

⁷ Although *Gray Box* is another plausible category seen as the blending of black box and white box, its presence in the literature is minimal. However, the work of Cedillo and Kieran (2003) promotes precisely this view and is discussed in detail in the next section.

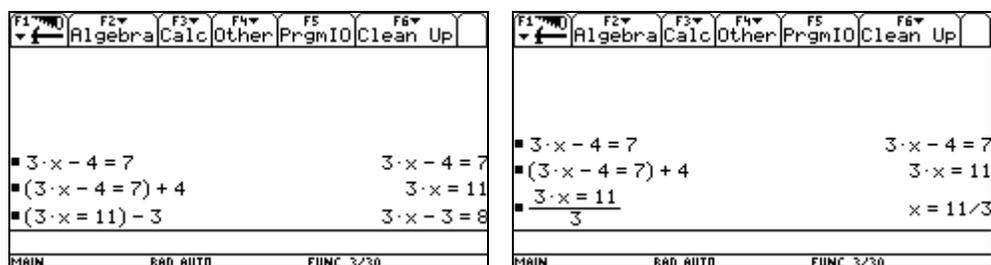


Figure 1.

In light of the above example, Heid and Edwards (2001) tout that it is CAS's ability to give instantaneous and nonjudgmental feedback that opens doors for novices who are struggling with concepts. Dick (2008) offers similar remarks with respect to the learning of calculus. While black box usage in evaluating $\int x^2 \ln x dx$ may benefit a student to some degree, a pop-up window in which the user must select a choice for u and dv (in an application of integration by parts) is far more likely to stimulate higher-level thinking.

Even in non-CAS settings (e.g., students working with the programming language *Logo*), the pedagogical use of computers has been shown to offer genuine learning opportunities. Hoyles and Noss (1992) discuss a particularly illuminating episode where a 13 year old student comes to understand that multiplication by a small number (between zero and one) decreases the original number. Although the authors mention several aspects that contribute to this learning, it is the immediate feedback from *Logo* that fosters refinements in the student's thinking and, consequently, adjustments to previous attempts to reach a goal. Conventional pencil-and-paper techniques seriously hinder this process. Consonant with the view of Heid and Edwards (2001), it is the immediacy and neutrality of the computer's responses that offer fertile ground for learning.

Amplifier

CAS can serve the role of amplifier to intellectual activity. That is, computer algebra tools can produce many varied examples in quick succession to the effect of offering assistance in discerning patterns and regularities which might otherwise remain hidden (Heid, 1997; Pea, 1987). It can also serve as a general experimental tool as one delves into the unknown world of mathematics. Generally speaking, such uses relegate “manual labor” (e.g., plotting points, repetitive multiplication, etc.) to a sometimes invisible level so that users may step back and generalize on a broader scale. This intrinsic attribute of “outsourcing procedures” to CAS is often equated with the amplifier role (Arnold, 2004; Heid, 1988, 1997, 2003; Heid & Edwards, 2001; Kutzler, 2003; McCallum, 2003; Palmiter, 1991).

The reader might find it surprising that there is little research to support amplification as putting students in a better position to learn mathematics. The reasons for this are two-fold. First, Pea’s (1987) original piece highlighted cognitive technologies in the broadest sense—programming languages, algebra systems, geometry software, and intelligent tutors. Vast amounts of the research literature cite amplification as an important use of CAS but this is almost always used to steer the discussion toward changing teacher practice or curriculum (cf., Heid, 1988; Palmiter, 1991). Second, the amplifier metaphor tends to be particularly well-suited to students’ generalizations in graphing environments. For example, a student may graph three or four members of a family of functions and formulate conjectures with respect to the changes on screen. Thus, the halfhearted attention this has received can be attributed to standard graphing calculators and their ability to perform these functions just as well. Since such graphical

excursions minimize CAS's most prized possession—algebraic manipulation—this finding is not surprising.

With respect to the amplifier role, Kutzler's (2003) look into mathematical history provokes one to think about how amplification can bring inductive reasoning back to the frontlines of mathematics education. For example, in a time of rapid mathematical development, the ancient Greek view of mathematics was one of experimentation-application. Exactification (precision of proof) was only added later leading to the cycle experimentation-exactification-application. The Bourbaki (French) system emphasized mainly exactification-application and this eventually morphed into modern day mathematics. This process of learning mathematics—stripped of the experimental and inductive components—is still the current paradigm but leaves much to be desired. Does CAS have the potential to bring this experimental component back?

Discussion Tool

The externalization of mathematical ideas to foster dialogue in classroom settings is a mainstay in CAS research (Guin & Trouche, 1999, 2002; Heid, 1997; Pea, 1987). Pea (1987) asserts that cognitive technologies “make *external* the intermediate products of thinking . . . which can then be analyzed, reflected upon, and discussed.” (p. 91). A nice example of this functionality can be found in the work of Pierce and Stacey (2001). This study examined 30 students in Australia as they took an undergraduate course in calculus in which the CAS *Derive* was integrated. Although the researchers' aims included examining students' flexibility in representations through amplification, the authors were particularly interested in whether CAS prompted meaningful mathematical discussions. When students were asked about whether conversations took place while

sharing a computer, 74% of the responses were either “very often” or “always.” A student’s perspective on this issue is especially illuminating: “In the computer labs we get together as a group. Something will happen on one machine and everyone will go and look and talk about it.” (Pierce & Stacey, 2001, p. 37). Although it is questionable whether such discussions were truly meaningful, the authors cite the growing body of CAS research as evidence of the computer’s potential to initiate discussions about what appears on screen (Boyce & Ecker, 1995; Drijvers, 2003; Kieran & Drijvers, 2006).

Using a more direct approach to spark discourse, Guin and Trouche (1999) examined students’ development/evolution of strategies with a novel physical classroom arrangement. Each day, a different student’s CAS calculator (TI-92) was connected to a large projector for the whole class to view (even though every student had his/her own calculator). This special student, called the “sherpa student,” played a central role in the lesson by assisting the teacher with lesson content and syntactical issues for his/her classmates to view. It is mentioned that this format fostered an environment of open discussion and debate in two notable ways. First, the small calculator screen—often personal and private to the user—was on public display for discussion. Often the dialogue was multifunctional in addressing mathematical, syntactical, or otherwise peculiar aspects of CAS. Second, the unique physical arrangement defined and reinforced social norms of mathematical activity that were predicated on free and open dialogue. Although this research has played an important role in sparking additional CAS-related studies, many of the details in this article are omitted but cited elsewhere (Guin & Delgoulet, 1996). Additional contributions (e.g., Boyce & Ecker, 1995) provide evidence that CAS can be especially fertile in promoting meaningful discourse, even in

cases where the teacher is the sole user of CAS. This paper is discussed in a later section concerning the timing of CAS implementation.

Catalyst for Reform

Broadly speaking, reform in education might be viewed as any movement that results in a nontraditional approach to learning a subject, irrespective of whether change occurs through teaching or student activity. Because CAS's impact on mathematics curricula and teaching are the foci of the final section of this paper, this section will simply sketch the main ideas. Research that highlights this transformation explicitly includes the works of Heid (1988) and Palmiter (1991). For example, Heid (1988) utilized a "concepts first" curriculum in which a group of students used CAS in the learning of calculus concepts, postponing skill-oriented mastery until the final three weeks of the semester. Meanwhile, a control group learned calculus in the traditional sense of blended skills and concepts. The results of the study showed no significant difference between the groups with respect to procedural mastery. This outcome *directly* challenges the assumption that procedural fluency need precede conceptual fluency in the calculus curriculum. Even while this assumption may not be explicitly embraced by mathematicians or teachers, it is deeply woven in the fabric of the K-16 curriculum.⁸

Similarly, engineering calculus students who learned the subject through the CAS MACSYMA significantly outperformed students who learned the subject in a traditional manner (Palmiter, 1991). This variation occurred both on conceptual and traditional assessments. The results again point to the potentials of computer algebra as a central

⁸ A common illustration of this is seen in "intermediate algebra" preceding "advanced algebra." The former emphasizes procedural fluency with little attention to meaning while the latter stresses concepts and connections across different representations.

player in mathematics classrooms, especially with the promising finding of equal or greater success rates for those learning with CAS. More details from each of these studies are provided in the section titled CAS: Curriculum and Instruction.

Finally, in a study of the use of *Maple* in remediation, Hillel et al. (1992) remark on the necessity of making sequential changes (and omissions) to the course in order to accommodate for the presence of CAS. Specifically, due to the wide array of situations that *Maple* treats uniformly, the authors found congruence in using a general approach to teaching functions. This clashes with the traditional hierarchy of first introducing lines, followed by quadratics, then polynomials, etc., as would be considered standard in mathematics curricula: "...a student using *Maple* can analyze $x^2 \sin x$ just as easily as x^2 if taught what aspects of the behavior of functions are useful to look for." (Hillel et al., 1992, p. 136). Additionally, this research coheres with other studies (Heid, 1988; Judson, 1990; Palmiter, 1991) which emphasize (a) learners' directive focus on conceptual and interpretive aspects of mathematics and (b) an atmosphere more conducive to experimentation and conjecture (cf. Kutzler, 2003).

A Model of CAS Use

Given the variety of uses discussed in this section, it is helpful, if for organizational purposes alone, to rank the spectrum of CAS utility in a way that synthesizes this multiplicity. For example, the literature reveals that black box usage offers little to learners' conceptual growth but it is precisely for this reason that this use is both uncomplicated and commonplace. In general, it appears that computer tools used this way pose a minimal threat to the "traditions" of mathematics teaching since it serves chiefly as a secondary authority figure. Quite the contrary, using CAS to reshape

mathematical activity/pedagogy requires greater innovation in concert with students' emerging cognitive needs. This role redefines the status quo, invoking in some an allegiance to specific CAS uses or in others, a profound skepticism toward the loss of "classical content." Embracing the view that something may be gained from each of the uses discussed here, a rudimentary model is proposed below.

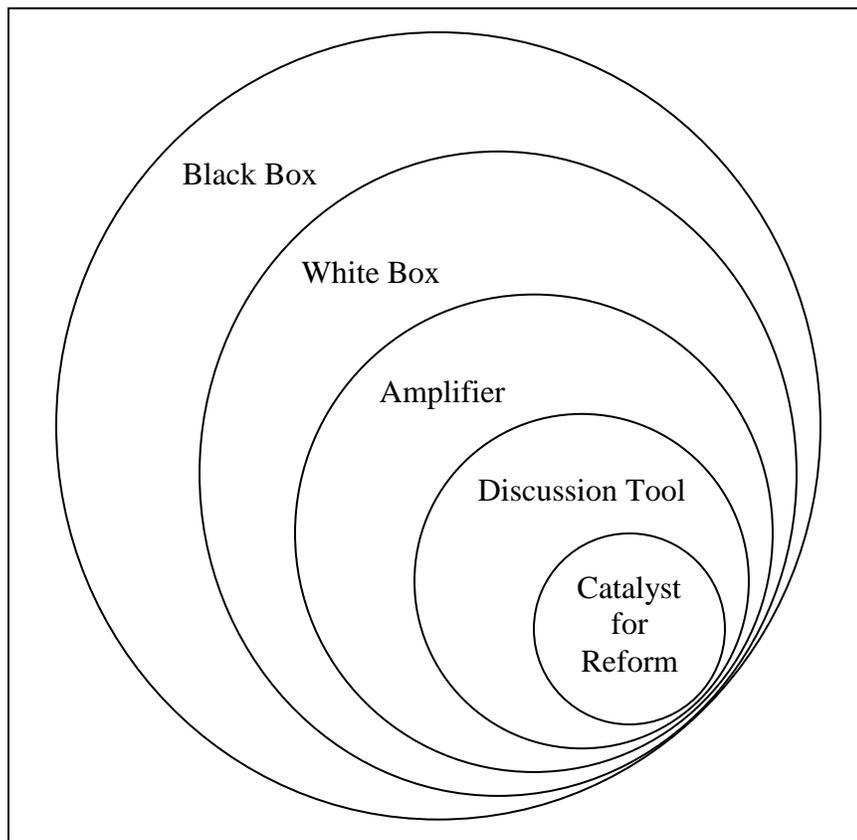


Figure 2.

The nested model was chosen for the simple reason that CAS use at any level likely subsumes its less sophisticated uses. For example, using CAS as amplifier takes advantage of both pedagogical tool (white box) and black box (de Alwis, 2002). On the other hand, using CAS *solely* as a black box may not—in any conceivable way—incorporate any of the other uses captured in this section (Buchberger, 1989).

Additionally, the sizes of the circles in Figure 2 are meant to convey—albeit crudely—both the category’s ease of implementation and degree of presence in CAS classrooms. Generally speaking, successively smaller circles signify decreased popularity of such use—most likely a function of the thoughtful purpose and investment needed to make this a classroom reality. At some point in the future, it would be interesting to investigate how learners might interpret the influence of these uses on their knowledge of mathematics, as well as the specifics of instrumental genesis in the adaptation of such uses.

CAS: WHEN TO IMPLEMENT?

Is there an appropriate or optimal time to introduce powerful CAS in the learning of mathematics? The very nature of this question is fraught with many possible interpretations. First, the reader might assume the context to be a specific course (e.g., algebra) and then ponder where CAS might fit. In contrast, one might view this from the broader perspective of an individual’s compulsory education (Gordon & Kleinstein, 2002; Pea, 1987) and from this, assert an appropriate entry point (e.g., high school). In short, there exists such a high degree of plurality in the literature that an exhaustive categorization of this sort results in little value added. Such diverse examples include CAS and/or computer tools as (a) media to teach concepts before skills (Heid, 1988; Judson, 1990; Palmiter, 1991), (b) enrichment tools to be introduced at a young age (Heid et al., 2002; Hoyles & Noss, 1992), (c) mediators in abstracting or formulating mathematical primitives (Hoyles & Noss, 1992; Monaghan & Ozmantar, 2006), or (d) facilitators in remediation (Hillel et al., 1992; McCallum, 2003; Schultz, 2003). Suffice it to say, this multitude of perspectives represents only a small snapshot of the literature.

The definition of “time” in this paper will be a refinement of the first scenario proposed above. That is, in the course of learning almost any material at any level of schooling, teachers will surely be granted opportunities to teach “new” mathematics to their students.⁹ It is in the conjoining of new mathematical concepts and powerful CAS where the learning experience may be enhanced (NCTM, 2000). Armed with a computing tool that can effortlessly execute the “raw skills” from such concepts, we then ask *when should CAS enter the learning experience?* In this section, three powerful examples across different levels of schooling—middle grades, secondary, and college level—will illustrate a variety of perspectives on timing. No association between the variables “grade level” and “timing” is claimed although one could surely speculate on the issue. Differences aside, the studies converge on one important aspect: CAS is used as a pedagogical tool in aiding creative thinking, quality investigation, and purposeful discussion, all to a degree that appears less likely in the absence of CAS.

To begin, Cedillo and Kieran (2003) report on a refreshingly interesting study of CAS use (TI-92) in grades seven through nine spanning 15 different schools in Mexico. The authors begin by providing a brief background of common findings and perspectives on CAS research. First, a great number of CAS studies focus merely on the transformational aspects of algebraic activity—that is, solving equations, factoring expressions, and numerical solving. Such restricted use appears to pigeonhole CAS’s role as a means to an end (i.e., obtaining an answer). Second, there is no shortage of disputes questioning the need for learners to possess meaningful understanding of concepts prior to utilizing CAS as a black box (Ball, Pierce, & Stacey, 2003; Drijvers, 2002; Pierce & Stacey, 2002). Cedillo and Kieran (2003) claim that this image “polarizes

⁹ “New” here refers to something which the students are seeing for the first time.

the discussion in terms of ‘before’ and ‘after,’ with pupils using the black box simply as a tool for problem solving *after* they have learned to reason about the problem solving situation. Implicit in the argument is the limited view of CAS merely as algebraic tools . . .” (p. 221). Given this backdrop, the primary goal of this study was to explore how CAS could facilitate genuine *learning* of algebra. The authors call this a “gray box” environment—an intermingling of the black box and white box approaches.

As underlying theory, the authors borrow heavily from Bruner’s work on children’s development of language (Bruner, 1983). Given that children learn language without explicit instruction of syntax or grammar, the authors embrace the view of “algebra as language” or “linguistic code” to investigate whether students could learn algebraic ideas without formal instruction of nomenclature (including definitions and theorems). The authors further exploit Bruner’s research in the development of mathematical activities that have puzzle-like qualities; after all, Bruner found leaps of advancement in children’s language development while immersed in situations of play.¹⁰

The authors discuss the collection of activities designed to align with the above theory. The activities consisted of six blocks of content (e.g., pattern development of numbers, equivalence of expressions) while each block contained anywhere from 10-15 worksheets. The researchers offer a close look at four samples to inform the reader of the nature of such activities. It is clear from the examples that students were to conjecture about numerical relationships by studying tables, writing computer programs, reflecting

¹⁰ As an aside, Goldenberg (2003) makes references to ongoing debates in bilingual education. Some argue that intellectual development should not be hampered by lack of English utility so some learning may need to take place in the native tongue. Others believe that students should not be denied the full exposure to English. Perhaps a gross oversimplification of an extremely problematic phenomenon, Goldenberg makes clear that the language these students need is English. So can CAS grant access to important *ideas* before algebra (the language) is fully developed?

on their actions, revising if need be, and explaining their findings. Given the informal but curiously game-like nature of these activities, the reader may well infer that—in the eyes of the students—“variables” were objects that stored values. Similarly, “expressions” were computer programs that students wrote, compiled, and executed.

The data used for reporting came from all of the participating eighth-grade classes. Data were collected over a three-year period and consisted of analysis of videotaped classrooms, field notes from classroom observations, and interviews with students. No additional information is provided but the authors cite another study from which additional details are offered (Cedillo, 2001). The research findings are mostly drawn from asking the students two questions on a cyclical basis: (1) What does the letter (variable) mean to you? and (2) What do the programs (expressions) that you write mean to you? The representative quotes that are provided in response to these questions bear a striking parallel to the modern conventions of “algebraic variable” and “algebraic expression” respectively. The researchers use student responses as evidence to support their assertions that (a) students can learn the fundamental ideas of algebra by simply *engaging* in algebraic activities (consonant with Bruner’s work), (b) students may make meaningful connections between algebra and arithmetic by way of grasping the flexibility of variables, and (c) students come to view expressions as devices that will compute a value for any desired input.

Their boldest claim is that the CAS experiences/activities induce and support students’ reasoning about the programs they choose to write. Although this seems a bit of a self-fulfilling prophecy, the authors offer evidence of students’ idiosyncratic approaches to writing programs. For example, this variety is worth noting in an activity

in which students were asked to generate the expression for the ‘number of white squares’ in subsequent diagrams given several diagrams early in a sequence (Figure 3).

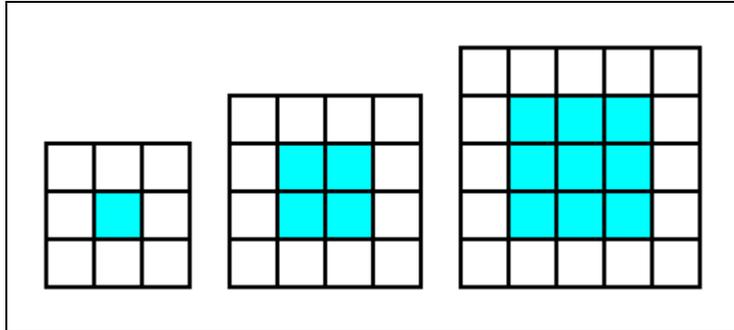


Figure 3.

Students’ methods of solution included table construction, direct computation followed by compensation (what the authors called a “shifted list”), and geometric approaches (subtracting the shaded area from the total area). The results $4(k+1)$, $k \cdot 4 + 4$, and $(a+2)^2 - (a \cdot a)$, appear different but are algebraically equivalent.

Although this study seems to hold great promise in the development of algebraic understanding in a CAS setting, there are three fundamental drawbacks. First, there is no discussion of difficulties encountered by the students in using the TI-92. This is most puzzling given the vast research on instrumentation of CAS. Were the students competent users of the tool from the first day? Were they somehow liberated of the obstacles omnipresent in CAS learning (Drijvers, 2000, 2002)? Both seem highly unlikely. Second, there is no information about the validity or reliability of the instruments used in this study. For example, the reader is told that “...materials were developed, teachers were trained, ongoing support was provided...” (Cedillo & Kieran, 2003, p. 222) but the general lack of detail makes it difficult for the reader to judge the quality of the instruments used and the training provided. Third, the worksheets taken

from the blocks were used a minimum of two times per five meetings (per the request of the researchers). If, in fact, this was the reality, the reader can only wonder what was happening the other three days. Were the students learning “algebra” in the traditional sense? In what kinds of activities were the students engaged and how did this affect their learning of algebra in this nontraditional sense? The answers to these questions are important and may render some of their findings questionable.

Embracing the view of the co-development of mathematical knowledge and technical CAS skill through instrumental genesis, Kieran and Drijvers (2006) describe a three-year long research program with the aim of describing how students’ thinking in a CAS environment is influenced both by technology and traditional means (paper-and-pencil techniques) and how students’ techniques, abstractions, and proof methods intertwine and co-emerge with the task at hand.¹¹ In alignment with the purpose of the study, the authors planned to measure the epistemic role of such experiences and, consequently, give a decisive value to the techniques developed by students in the CAS/pencil-and-paper medium. In doing so, they utilize the anthropological “theory of didactics” as presented by Chevallard (1999). This view may be briefly sketched as *knowledge development through practice*.¹² Kieran and Drijvers (2006) confer that this

¹¹ There is research predating this work (Heid, Hollebrands, & Iseri, 2002) that illustrates a similar phenomenon. Kevin, the subject of this research, reasoned a fair amount on paper and on computer in the context of a challenging mathematical exercise. The interplay between his deductive reasoning and use of technology to explore/verify is captured in this paper. Furthermore, Guin and Trouche’s (1999) study of students’ work methods describes the “resourceful work method” as one in which calculator potentiality, theoretical work, and pencil-and-paper techniques intermingle in a web-like fashion to influence user actions.

¹² The overarching theme of this “theory of didactics” is the multifaceted theoretical construct of *praxeology*, describing, in this case, mathematical activity. This praxeology consists of four components: (a) *tasks* (of a similar nature) which enrich meaning for the learner, (b) *techniques* which assist in completing the tasks, (c) *technology* as a means of validating the techniques, and (d) *theory* as discourse systematization that supports such practice (Chevallard, 1999). As some assert, this epistemology distances itself from that of constructivism (Artigue, Batanero, & Kent, 2007).

theory served as a means for developing the tasks as well as gathering and analyzing the data for the study.

During the first year of the study, the research team focused on task development. Two areas were investigated in depth: (a) making mathematical connections between solving equations and equivalence of expressions, and (b) pattern development and proving in the context of factoring. The authors chose areas deliberately unrelated in order to illustrate the robustness of the dialectics between technique and theory. Two tenth-grade classes, each with 17 students, were the sources of data for each of the areas explained in the study. To convey the progression of the participants' knowledge development at a sufficient grain size, the research team collected an abundance of data including (a) classroom observations of student-teacher interactions using two video cameras (one camera captured specific interactions while the other filmed the classroom at large), (b) researcher's field notes, (c) individual and pair wise audio-taped interviews with students (to clarify ambiguities), (d) pretests and posttests of students' knowledge, and (e) student activity sheets and responses to tasks. The final year of the study was dedicated to synthesizing and analyzing this data.

For both units of study (algebraic equivalence and factoring/proof), the authors provide the reader with the details of the objectives of the units, examples of student tasks, and representative reactions/responses to such tasks (including student-to-student dialogue). One particularly illuminating account is one in which the emergence of symbol sense, pattern seeking, generalization, and proving unfolds for the students. Important here is that CAS often displays factorizations that vary with what students produce by hand. This initially generates confusion but sometimes leads to new

conjectures and theories in light of the differences. Two aspects were found to be critical in fostering students' willingness and persistence to advance with these tasks: (a) the intellectual need to resolve differences in pencil-and-paper work versus CAS output, and (b) a newfound awareness of structural differences in expressions not previously encountered. With this, the researchers capture the students grappling with results in which they had to coordinate existing knowledge with new findings often embedded within previously established theories. These realizations provided a vehicle for new techniques that students used to test and validate their conjectures, suggesting high epistemic value. The ultimate finding here is that the techniques, tasks, and theoretical aspects of student thinking contain elements from both CAS and pencil-and-paper environments, so revealing the difficulty in isolating one factor as the sole force for mathematical learning.

Finally, the work of Boyce and Ecker (1995) illustrates nicely the potential benefits to students who experience CAS at the undergraduate level *after* a mathematical concept is learned.¹³ The heart of the paper includes two examples that illuminate such benefits, the second being a strikingly vivid example of how pedagogy may be drastically altered in the presence of CAS. What starts as a fairly routine computer exercise—the instructor leading the class by using *Maple* to find the first 10 Taylor polynomials to $f(x) = \sin x + e^x$ on the interval $[-6, 2]$ centered at $c = -2$ —quickly turns into a far more meaningful engagement. A student asks why the midpoint was chosen and whether this would provide the “best” Taylor polynomial given a fixed degree. What follows is a most unusual classroom discussion of what was meant by *best*, a negotiation and eventual

¹³ Although this work is predominantly anecdotal to the point of powerful story telling, it is included here for just this reason—its swaying power and thick description (Gall, Gall, & Borg, 2007).

consensus that one should minimize the area between the function and its approximation to deem it *best*, and the full exploitation of *Maple* to explore this line of inquiry. In developing an error term to monitor this phenomenon, *Maple* generates an expression containing 130 terms. Although many students were surprised by the expression, the class as a whole was able to see past this difficulty realizing that a plot could be generated and the typical techniques of calculus could be applied to find the location that would minimize the error function.¹⁴

Most intriguing here is that the student asked a question likely motivated by the instructional setting and tools available. Specifically, the CAS environment allowed the instructor and students an opportunity to investigate and construct a tentative solution to this problem—one that would be far too cumbersome by traditional means. The authors say it best:

This discussion of Taylor polynomials was at a much higher conceptual level than any of our previous lectures on the topic. The interactive and spontaneous nature of our discussion would not have been possible without a computer algebra system. The much deeper understanding of Taylor polynomials gained that day by these students was due, in part, to the use of *Maple*, which allowed them to see past the details into the conceptual heart of the subject.” (Boyce & Ecker, 1995, p. 49).

Lastly, this example offers evidence of CAS’s ability to perform “outsourced” tasks effortlessly—granting learners face-to-face opportunities with mathematical meaning.

From the studies above, it appears that using CAS before or during the conception of a mathematical principle carries several advantages. These include (a) making abstract concepts more accessible (Hillel et al., 1992), (b) dwelling in the novelty of play in learning with computers (Cedillo & Kieran, 2003; Hoyles & Noss, 1992; Pea, 1987), (c) offering numerous scenarios in quick succession for generalization (Arnold, 2004; Boyce

¹⁴ In a similar manner, Heid et al. (2002) describe “Kevin” as another student who could see past similar oddities. As the authors claim, “the ability to recognize when to ignore the results of technology and pursue alternative strategies is a quintessential feature of informed use of technology.” (p. 213).

& Ecker, 1995; Pierce & Stacey, 2001), (d) granting increased flexibility in problem solving (Cedillo & Kieran, 2003; Heid, Hollebrands, & Iseri, 2002; Lagrange, 2003; Pea, 1987; Ruthven, 2002), and (e) providing a set of referents from which to induce future knowledge (Kieran & Drijvers, 2006; Schultz, 2003). As an aside, one might further argue the transcendence of the implicit message that computers are part of modern day research mathematics. Concrete evidence of this can be found in the work of Appel and Haken (1976) and Wolfram (2002).

On the downside, there are conflicts with integrating CAS at this critical time of mathematical learning. Concerns here include (a) the context-bound knowledge that students are likely to develop and the question of knowledge transfer outside of this context (Artigue, 2000; Hoyles et al., 2004; Lagrange, 1999; Ruthven, 2002), (b) the arduous process of instrumental genesis (Artigue, 2000, 2002; Guin & Trouche, 1999), (c) obstacles and other mystifying incongruities (Bossé & Nandakumar, 2004; Drijvers, 2000; 2002; Gordon & Kleinstein, 2002; Lagrange, 1999; Pierce & Stacey, 2002), (d) the consequential effects of weak conceptual understanding and its manifestation in the CAS environment (Drijvers & van Herwaarden, 2000; Drijvers, 2003), (e) the fact that CAS does not, by itself, stimulate deep thinking (Guin & Trouche, 1999; Hoyles & Noss, 1992), and (e) the degree of fidelity and accuracy with respect to CAS's natural limitations (Dick, 2008; Goldenberg, 1988). Many researchers cite one or several of the above issues as legitimate concerns to address as we move toward a more technology-integrated classroom.

In the work of Boyce and Ecker (1995), it is difficult to deny that such an experience enriched and augmented the mathematical understanding of the students.

However, this particular study did not have students engaging in the use of CAS even if this is the common practice at Rensselaer. In other words, students were simply observing the powers of CAS unfold before their eyes. Given this, the concern lies in the implication of relegating CAS to the “post-learning” of mathematics as Cedillo and Kieran (2003) stress. Students, teachers, and administrators alike may come to believe that calculators/computers add little to the *educational* experience. Although such tools are deemed legitimate when embraced through a social or scientific lens, this legitimacy is significantly reduced when attempting to assess whether such devices improve mathematics teaching and learning (Artigue, 2000). Regardless of one’s perspective, to implement CAS in one’s classroom is one battle, but deciding *when* to do so in the trajectory of student learning is another battle that has just begun.

CAS: CURRICULUM AND INSTRUCTION

For more than 20 years, attention has been drawn to the potential impact of CAS in reshaping mathematics curriculum and promoting change in teacher practice. Without a doubt, the presence of powerful technology has caused an imbalance to the well-formed traditions of teaching a curriculum built on the absence of technology (Lagrange, 1999a). Given the new possibilities afforded by CAS—as amplifiers, experimental tools, and catalysts for change—one finds new choices in what to teach as well as how to teach it. As Goldenberg (1988) purposefully states, “having the choice obliges us to make it.” (p. 166). The purpose of this section is to provide a glimpse into how CAS has infiltrated the mathematics curriculum, sometimes resulting in radically reformed views of how best to teach a mathematics that harnesses this power.

Curriculum

With respect to the content that students learn, the research studies discussed here fall into one of three general categories. First, some emphasize the natural potential of CAS to free students from exercises of a drill nature, leading to opportunities to connect mathematical ideas. In general, these studies place an emphasis on how the traditional progression in mathematics courses may be resequenced in light of computer technology (Heid, 1988; Judson, 1990; Palmiter, 1991). Second, there are studies that focus on the nature of mathematical content and claim that CAS may lead educators to look for different ideas and ask different questions of students (Lagrange, 2003; Pierce & Stacey, 2002). Finally, there is research that utilizes CAS in innovative ways to teach topics deemed critical to one's mathematical upbringing (Ayers et al., 1988; Cedillo & Kieran, 2003; Zbiek & Heid, 2009). Based on an assessment of student learning, researchers often state their case for mainstreaming these experiences into curricula and professional development for teachers. These studies are discussed presently.

Several important publications from the late 80s and early 90s shared the burden of revealing the nature of learning in CAS environments (Heid, 1988; Hillel et al., 1992; Judson, 1990; Palmiter, 1991). The work of Heid (1988) is prototypical in the sense that it presented the possibility of reversing concepts and skills in the teaching of business calculus. This resulted in not only increased conceptual understanding but showed minimal effects on procedural fluency. This finding was strengthened in later years by similar studies (Judson, 1990; Palmiter, 1991). Based on the assumption that instruction with attention to meaning could have lasting effects on learning, Heid conducted a study in which two experimental groups ($n = 39$ total) learned the concepts of calculus in the

first 12 weeks of the semester via the CAS *MuMath*. It was only during the final three weeks of the class that related procedures/algorithms (taking derivatives, evaluating limits, sketching curves, etc.) were demonstrated to the class with an emphasis on skills carried out by hand.¹⁵ In contrast, a large lecture-based section of calculus ($n = 100$) was used as a control group. This group learned calculus in the traditional sense of learning rules, procedures, and algorithms in conjunction with the principles and ideas of calculus. Both groups took identical final exams.

The purpose of Heid's work was to determine whether the experimental groups could learn calculus concepts without previous exposure to accompanying procedures. The author provides a few samples on how the computer was used as well as a snapshot of the interface of *MuMath* (which appears quite similar to modern day versions of *Maple* and *Mathematica*). Consonant with much of the literature, the use of *MuMath* externalized the mathematics and afforded students opportunities to discuss important ideas. Moreover, flexibility as an amplifier allowed for the generation of many examples to explore concepts and analyze difficult problem scenarios. To address the research question, Heid gathered information from multiple sources including audiotapes from most of the experimental classes, classroom observations from the control group section (the lead researcher was the teacher in the experimental group), interviews with 20 students (15 from the experimental groups, five from the control group), and examination results from both groups. Audiotapes from the experimental groups' classroom discussions and student interviews were transcribed and studied. Additionally, classroom

¹⁵ Heid mentions that the two experimental sections were treated in a slightly different manner. In one of the sections, the teacher introduced very basic algorithms connected to the concepts learned. In the other section, algorithms were delayed until the final three weeks of the semester.

observations produced field notes while some students offered their personal notebooks at the researcher's request.

The findings of the study are drawn primarily from three sources—student interviews, written assessment of conceptual ideas, and final examination results. Overall, striking differences were found between the two groups with respect to their descriptions of calculus ideas. For example, in response to the question, “What is a derivative?” the disparity in student responses across the experimental and control groups is captured below:

A derivative is a number of things. Mathematically it's a slope—but I think of it as a rate of change—a description of change. In math, the symbolic language we've been using, we've been working in two variables. The derivative is a description of the change in one versus the other—at a certain point—at a certain time—at a certain place. (Heid, 1988, p. 15, quote from experimental class student).

They explain it as being what the curve is doing—and I don't see the curves. I can think of it as how fast the curve is going . . . If it's positive, I keep thinking it's definitely going somewhere—if it's zero, it's not—if it's negative, it's going the wrong way. (Heid, 1988, p. 15, quote from control class student).

One discerns a far richer and mathematically accurate explanation from the former remark as well as a profession of ownership of the derivative idea. As Heid (1988) conveys, this was not an anomaly but rather representative of the experimental group as a whole. The second finding is drawn from the results of a concept test given prior to the final three weeks of the semester. As reported, the experimental group outperformed the control group on 14 of 16 parts of the test—in some cases by very large margins. Finally, final examination scores accompanied by descriptive statistics were reported for the two experimental groups (Experimental group 1: $n = 18$, $M = 105$, $Med = 105$; Experimental group 2: $n = 17$, $Mean = 115$, $Median = 127$) and the control group ($n = 100$, $Mean = 117$, $Median = 116$). Although the author reports the mean scores as “not substantially different” (Heid, 1988, p. 18), nothing beyond this is displayed for the reader.

In looking back at this work, one does not find it surprising that the experimental groups displayed a superior command of calculus knowledge both in expression and in written assessment. After all, these students experienced a curriculum that allowed them free access and exploration of ideas through *MuMath*. Moreover, the questions on the conceptual test mirrored quite closely the experiences that the students in the experimental sections encountered. What bears educational significance is that these students—even with instruction of the rote processes in calculus as an afterthought late in the semester—still fared well on skill-oriented problems while the control group spent the entire semester practicing these skills. Given this finding, it urges one to ask why mathematics curricula continue to focus on the computational aspects of the subject when it is clear that a curriculum as described here not only brings about a deeper grasp of concepts but a nominal decline in procedural fluency.

In a similar study of engineering calculus students, Palmiter (1991) found that students using the algebra system MACSYMA ($n = 39$) outperformed students learning calculus in a traditional lecture setting ($n = 39$). This imbalance occurred on multiple assessments. In the MACSYMA group, CAS was used as a super calculator to compute limits, derivatives and integrals—that is, absorbing the procedural aspects of mathematical activity. Given this format of instruction, the CAS group covered the same core material as the control group but in half the allotted time. At the end of the instructional period (five weeks for the experimental group and 10 weeks for the control group), identical computational and conceptual assessments were given. Each of these examinations received balanced input from the instructors and teaching assistants from each group and agreement was reached that both sections addressed the material pertinent

to successfully answering the exam's questions. On both assessments, there were significant differences in performance that favored the MACSYMA group (Conceptual Exam: CAS group score = 89.8 [n = 39], Control group score = 72.0 [n = 39], with $T^2 = 1.20$, $p < 0.001$; Computational Exam: CAS group score = 90.0 [n = 38], Control group score = 69.6 [n = 39], with $T^2 = 0.92$, $p < 0.001$). Although the higher achievement of the CAS group suggests a deeper understanding of calculus concepts, the author is quick to point out that both the Hawthorne effect and teacher effect are possible confounding variables to drawing this conclusion.

Myriad studies focus on other aspects of the mathematics curriculum different from the issues emphasized above. For example, Lagrange (2003) discusses an interesting scenario in which the power of CAS may drive educators to ask different questions possibly leading to mathematical connections not emphasized in standard curricula. (This discussion can be found on pp. 8-9.) Parallel to this work, Drijvers and van Herwaarden (2000) and Drijvers (2003) discuss the advantages of how CAS may deepen students' understanding of parameter by moving from concrete problem situations to more generic/abstract settings in which variables play the roles of numbers. As Goldenberg (2003) stresses, in lieu of the purely mathematical ideas we emphasize in school settings, educators might start thinking about the functionality of CAS and how this may alter the landscape of mathematical learning. He suggests that we emphasize how to enter problems into CAS, what questions to pose (see also Arnold (2004)), what to look for, and the like. Consonant with this view, Pierce and Stacey (2002) provide an interesting example of a newer kind of understanding that needs increased attention in a

CAS curriculum if computers are to play a successful role in mathematical learning. This is discussed presently.

Pierce and Stacey (2002) discuss a certain type of algebraic understanding they deem necessary for effective CAS use. This understanding, called *algebraic insight*, is defined as “that part of symbol sense needed for finding a mathematical solution to a mathematically formulated problem.” (p. 623). They argue that teachers should shift their focus away from obtaining correct answers to facilitating and monitoring students’ acquisition of such insights. They draw an interesting parallel to number sense in arithmetic—much like individuals estimate 113×46 to be close to 5000, an *algebraic* setting prompts users to anticipate answers of a certain type. This component of algebraic insight is called *algebraic expectation*. Here, students need to have a keen sense of mathematical conventions including order of operations, use of parentheses, and implied multiplication in both handwritten and CAS activity. Similarly, effective use of CAS demands familiarity with mathematical structure (e.g., equivalent factored forms) and recognition of salient mathematical features (e.g., $e^{2x} + 6e^x + 7$ is quadratic in e^x). A second component of algebraic insight is *linking multiple representations* that are displayed by CAS. Specifically, the authors remark on the “symbolic-numerical link” and the “symbolic-graphical link.” As an example of the former, the researchers argue that a student’s understanding of periodicity should guide him/her in numerical analyses of functions. For example, given a function such as $y = \sin(2\pi x)$, the act of setting a table at zero with integer increments provides a misleading table of values, reflecting a low degree of algebraic insight.

Finally, some studies suggest a reformulated view of the teaching and learning of mathematics through computer environments. The work of Ayers et al. (1988) provides an interesting account of how students' use of programming skills in a UNIX shell (similar to entering commands on CAS) shapes their understanding of composition of functions. The study hypothesizes that computer usage in the learning of composition can bring about the mental processes of *reflective abstraction* as described by Piaget (1976, 1978). Their central question concerns whether (and to what degree) computer experiences foster such mental processes. Because computers provide representations of entities that are both concrete and abstract, this setting is assumed to be particularly fertile in fostering reflective abstraction.¹⁶

To begin, the authors describe *shell scripts* and *pipes* as they are used in the computer setting. Shell scripts consist of commands entered into the computer to perform specific tasks (e.g., evaluating a function at $x = 4$) while pipes mimic the very act of composing two functions. In the study, there were two distinct groups—one labeled the “computer” group (which was further divided due to registration constraints) and the “pencil-and-paper” group. A pretest assessing understanding of functions and composition was given to the students and because of a few particularly high scores, some students were omitted from the study as the authors felt these individuals demonstrated mastery of the ideas under study. The authors offer little information beyond the five questions on the pretest being “standard” questions about composition while additional items were similar to what appeared on a posttest (described later). Aside from scoring these examinations in a consistent manner, nothing concerning the

¹⁶ On one hand, entities are concrete since they are visible (even if only in electronic form) but simultaneously, such entities are intangible artifacts.

validity or reliability of the assessments is discussed. In the end, the research followed 30 students—seven in the first computer group, six in the second computer group, and 17 in the pencil-and-paper group.

During the course of the six week class, the groups met weekly for a two hour block. The first and sixth sessions were reserved for testing (pretest and posttest). During the second and third sessions, the pencil-and-paper group was offered a fairly traditional exposure to function and composition as is typical in preparatory calculus. Meanwhile, the computer group used shell scripts and pipes to learn the analogous material. The authors provide examples of the similarities and differences in these exercises for the reader (even though it is unclear if the classes were actually observed). Both groups were encouraged to seek help from their instructors as the mathematical formalisms were not exposed in either setting. For the fourth session, all of the students met jointly to listen to a lecture on functions and composition. Finally, session five revisited the topics of the second and third meetings; here, the instructors made efforts to connect the material from the previous lecture to what was learned in the second and third sessions. Thus, of the six meetings allotted for each group, three varied so comprising the treatment for the computer group.

The posttest results are illuminating in that the computer groups mostly outperformed the pencil and paper group *irrespective* of whether the specific posttest question was considered in favor of computer usage (four questions), pencil and paper activities (four questions), or neither (seven questions).¹⁷ The authors' main claim is that “the computer experiences given to the students in the Computer groups were more

¹⁷ Although most of the statistics are not presented in the paper, the authors mention that they may be contacted for all of the data analysis.

effective in inducing the reflective abstractions involved in constructing the concepts of function and composition than was the traditional treatment given to the Pencil-and-Paper group.” (p. 256). This blanket conclusion is predicated on the fact that the posttest was “designed to indicate whether or not reflective abstractions involved in the construction of the concepts of function and composition actually took place.” (p. 253). Although one must question the validity of an instrument that assesses a learner’s ability to engage in reflective abstractions, this study adds to the knowledge of computer-assisted learning in two important ways. First, the students who participated in the computer activities *likely* constructed mental representations of composition that went beyond the mere view of mathematical formulas. Second, the computer treatment posed no observable negative effects in mathematical activities of a traditional nature. These findings resonate with the results from Heid (1988), albeit across different mathematical topics on different audiences with different computer software.

Finally, the work of Zbiek and Heid (2009) suggests how CAS may prepare students to study new mathematical content without being slowed by earlier mathematical deficiencies. Such a position challenges the status quo of a curriculum built on a hierarchy of progressive and sequential mastery. Their central thesis is that CAS can offer opportunities to grasp new content while repositioning previously learned material under the purview of “macroprocedure.” In this sense, students may focus on the novelties on hand by way of teachers using CAS to orchestrate the blending of new and old ideas. This, they argue, is often overshadowed in traditional instruction by technicalities from prerequisite knowledge.¹⁸ Two examples discussed are (a) the

¹⁸ Consistent with this view, Kutzler (2003) comments on CAS’s role as a means of *concentration*. By nature, mathematics almost always teaches a new skill while practicing an old skill—in some sense,

development of a reformulated view of finding a solution (e.g., a graphical solution may be of greater value than a complex algebraic solution), and (b) extensions of once established mathematical ideas (e.g., CAS may output $g(f(x)) = \sqrt{x^2} = |x|$ given $f(x) = x^2$ and $g(x) = \sqrt{x}$, often to the user's surprise). Although these ideas appear to hold great promise, the literature addressing obstacles of CAS suggests that students may fail to interpret a graphical solution as an "answer" or encounter widespread confusion due to the overt "black box" nature of CAS (Bossé & Nandakumar, 2004; Buchberger, 1989, 2002; Drijvers, 2000; McCallum, 2003). The casual nature of this report makes it difficult to weigh the merits of such a change in the mathematics curriculum.

To what degree does CAS change the face of mathematics? From the perspective of classroom learning and students in training, this change is drastic and cuts deeply across (a) the written curriculum (cf. Ayers et al., 1988; Ball, Pierce, & Stacey, 2003; Heid, 1988; Hoyles & Noss, 1992), (b) the wider-ranging aims and purposes of a mathematics education (Goldenberg, 1988; Lagrange, 2003; Pea, 1987), and (c) mathematics teaching (Pierce & Stacey, 2002). The research discussed here suggests a reorientation of goals with a heavy emphasis on conceptual ideas and principles (Heid, 1988; Judson, 1990; Lagrange, 2003; Palmiter, 1991). It can be inferred that deeper understandings are internalized if concepts are given explicit attention (Ayers et al., 1988; Cedillo & Kieran, 2003; Heid, 1988). The greatest pledge here is that exposure of concepts through computer activities proves beneficial to enhancing students' understanding while showing little effect on tasks of a traditional nature. As an outgrowth to this result, some researchers call for a close study of new issues in

interrupting the learning of the new skill repeatedly. CAS—it is argued—can allow full and uninterrupted concentration on the new topic.

mathematics *teaching* that arise from heightened use of CAS in problem solving (Drijvers & van Herwaarden, 2000; Lagrange, 2003; Pierce & Stacey, 2002). Issues relevant to classroom teaching are given attention in the next section.

Teaching

Researchers have been generally attentive to the effects of CAS in redefining teacher roles and responsibilities. The use of computers as described previously (in the curriculum and utility sections of this paper) suggests that the once embraced view of teacher as “conveyer of truth” is being questioned given CAS’s ability to illustrate important mathematical ideas. The research discussed in this section offers the current view that teachers might best orchestrate and facilitate students’ use of computer algebra. The authors warn however, that this responsibility is a difficult one to embrace, perhaps due to (a) teachers’ eventual need to relinquish what has been the traditional role of “telling” students mathematics (Garofalo, 1989; Schoenfeld, 1988, 1992), or (b) teachers’ desires to convey a perceived command of subject matter knowledge (Kendal & Stacey, 2001; Zbiek, 1995). Amid these tensions, the literature finds that teachers frequently resort to using CAS in diminished roles in order to align with their view of teaching mathematics in the absence of CAS. Using computer algebra in these ways (e.g., “black box” or “white box”) appears less likely to have an impact on student learning since its power is eclipsed by an allegiance to traditional means of knowledge acquisition.

A few of the studies cited earlier in this paper clearly propose changes in teacher roles in CAS classrooms. Guin and Trouche (1999) provide a good example of how instruction is reshaped in the sense that teachers facilitate the process of student instrumentation. Similarly, Hoyles et al. (2004) see the orchestration of classroom

activities as the teacher's chief responsibility in computer-driven settings. Both argue cogently that as students encounter mathematics through instrumental experiences, teachers need to actively examine and evaluate student thinking so that these constrained experiences become central parts of the mathematics that is learned. More than personal professions of theories of knowledge acquisition, it appears that these reports support the view that CAS can make mathematics the object of outward discussion and fundamentally be a catalyst for redefining mathematical activity. This position demands that the teacher play a role far removed from the all-knowing "distributor" of mathematical truths.

Perhaps the greatest concern the literature presents is that transitioning to this new role is far from easy (Kendal & Stacey, 2001; Pierce & Stacey, 2004; Zbiek, 1995, 2002). Although these studies embrace the less authoritative role of teachers in technology classrooms, they highlight the gradual (and often arduous) process to witnessing such a reality. For example, Zbiek (1995) reports on an experienced teacher's shift in views as she implements a CAS-driven algebra curriculum to secondary students. Zbiek sets the stage by informing the reader that the presence of technology infiltrates school mathematics with many changes—most notably a *mathematics* that is (possibly) unfamiliar to these seasoned teachers.¹⁹ As part of the larger CIME research study (Empowering Secondary Mathematics Teachers in Computer-Intensive Environments), the author first remarks on the training the in-service teachers received in experiencing the technology and curriculum firsthand. Also, there is a very brief discussion about the instruments/techniques used to collect data including six classroom observations, a series

¹⁹ See the previous section of this paper.

of 10 interviews spanning 13 months with the teachers, journal entries written by the teachers, and copies of classroom materials used in instruction.

The paper highlights the differences in one teacher's (LeAnne) espoused beliefs versus what actually occurred in her classroom. During the teacher's professional training, she showed a skillful fluency in using CAS—to generate examples, validate hypotheses, and solve complex problems. Moreover, she expressed her general preference for allowing her students to think and reason through mathematical situations in contrast to simply “telling” them the mathematics they should know. At the beginning of the 13 month observation period of the 60 teachers involved in the CIME study, LeAnne noted that the computer lab was a place to ask questions, struggle with meaning, and grapple with mathematics. By mid-year however, the researchers documented (through direct observation) a radical change in the way LeAnne handled the computer lab experiences with students. She now gave the students prescriptive worksheets with exactly the commands to carry out as well as sample worksheets so that students could “match” their work to the teacher's. In short, LeAnne saw the need to micromanage her students' discoveries. Zbiek notes that such a shift documents a tension in competing paradigms—how teachers come to see the role of CAS in the classroom versus how they feel the subject should be taught. The degree of discomfort that this teacher felt with the “mathematics” from the CAS is something worthy of in-depth examination.

The work of Kendal and Stacey (2001) examines how teacher privileging impacts instruction with CAS and, resultantly, what students learn.²⁰ Two teachers developed and revised a 20-lesson first-semester calculus program that incorporated CAS (TI-92) for

²⁰ Teacher privileging is a construct that attempts to describe how teachers teach, why they make the decisions they make, and thinking-in-action (Wertsch, 1990).

high school students. Classroom lessons were observed, audio-taped, and transcribed by the lead author. A journal of notes was kept current, teachers were interviewed, and students in the classes completed questionnaires and participated in task-based interviews. Although there is a fair amount of information offered in this study, there are questionable omissions (e.g., the reader never sees the interview questions asked). Additionally, in a comparison of competencies across the two classes, the reader is offered (a) only a partial view of the instrument used to assess student understanding and (b) no information on validity or reliability.

The overall findings give the impression that CAS did not change the way these teachers taught. Rather, the teachers incorporated CAS in a way that mirrored their overall philosophy about how one learns (or should learn) mathematics. For example, Teacher A who emphasized procedures/rules found CAS a very useful tool for outsourcing repetitive skills to generate patterns and establish rules. Furthermore, his misconception that CAS delivered exact answers (as opposed to approximations) motivated him to use CAS as a functional tool to enhance numerical and graphical sense of the derivative. In contrast, Teacher B focused heavily on conceptual links and mathematical meaning. In his lessons, he used CAS far less but he did so in a pedagogical manner—mostly to strengthen connections between the symbolic and graphical understandings of derivative. The message one takes from such work is that teachers of mathematics may see CAS as peripheral to the mathematical experience. Unlike advocates who may endorse CAS as a centerpiece in reform, these teachers used computer algebra in a casual and less than innovative fashion. This fact is only

magnified when viewed in light of recent research with computer algebra (Kieran & Drijvers, 2006; Zbiek & Heid, 2009).

Another study that dwells on teaching in a CAS environment is the work of Pierce and Stacey (2004). The study's primary aim was to explore whether students managed to overcome the technical challenges of CAS and, as a result, use it effectively. In doing so, it was found that "traditional" mathematical ability has no obvious connection to the principles of effective use, technical facility, or personal enjoyment of CAS. The findings as they connect to teaching are important. In teaching students to use CAS in meaningful ways, the teacher may be required to not only have an awareness of such issues but a sensitivity to the findings in this study. For example, assumptions such as "Kate is smart in math so she can use CAS easily" proved to be illusory.

Finally, Zbiek (2002) draws on several previously published studies of teachers and their use of CAS in mathematics classrooms. The underlying aim was to accumulate some tentative findings on factors that influence teacher behavior in CAS environments. Although the paper appears to be a collection of the author's hunches, she does an adequate job of conveying her point that teacher's views about *mathematics* and *teaching* strongly influence CAS utility in daily practice. Strong support of this was found in Kendal and Stacey (2001). Zbiek notes that the answers to the following three questions carry significant weight on teacher behavior with CAS:

- (a) How is school mathematics defined?
- (b) How is mathematical understanding defined?
- (c) How does CAS fit into day-to-day classroom teaching?

A few broad conjectures are drawn. First, teachers furnished little attention to the symbolic functionality of CAS in lieu of its graphical capabilities. Zbiek attributes much

of this to teachers' prevailing belief that mathematics should be done by hand. In fact, CAS was often used to merely verify results that were previously obtained by pencil-and-paper means. It is further mentioned that little effort was made to connect what was happening on the CAS with traditional pencil-and-paper work. Zbiek proposes that state and national assessments may direct teachers' attention away from technology, reaffirming power to analytic methods.

Second, the teachers held a somewhat impoverished view of "understanding" mathematics—that is, understanding is consonant with being able to carry out algorithms precisely. This partially explains why some of the teachers used CAS as they did—to verify previously computed results and to execute specific keystrokes to carry out unambiguous mathematical tasks. Third, since CAS modifies the role of teachers by placing them into less authoritative positions, teachers lose some degree of comfort and control. This partly explains the actions of using CAS in a closed and narrow fashion. As we are told, "incorporating CAS changes the face of the classroom . . . classroom comfort quickly vanishes as CAS use both permits and requires alternative decisions, actions, and understandings." (Zbiek, 2002, p. 135). Assigning tasks that are both free of ambiguity and assess understanding in small pieces allows the teacher to regain some of this control.

As this section illustrates, instructor roles in CAS classrooms may take teachers to unfamiliar territory. Quite predictably, teachers gravitate toward a more comfortable reality but often at the expense of using CAS in a pointed and directive way. An example that pervades the literature is the previously discussed 'black box'—a use that merely echoes previously learned mathematics but with no attention to meaning. A competing

problem may be that teachers have ways of viewing the world and teaching their subject matter; the sophisticated uses of CAS may/may not align with such views.

It is interesting to note that research on how CAS influences the teaching of mathematics is available but slim (Heid & Blume, 2008; Lumb, Monaghan, & Mulligan, 2000) but research on teachers' struggles to implement technology appears to be more developed (Zbiek & Hollebrands, 2008). Regardless, these studies converge on one central issue—CAS is such a powerful learning tool that we must consider its role in mathematics classrooms and how teachers can best adopt it. In general, the literature indicates that teachers may need to bolster their confidence with personal use of CAS prior to achieving an adequate comfort level in teaching mathematics shaped by CAS. Consonant with the view of Zbiek and Hollebrands (2008), learning to teach with technology is a developmental process whose eventual success is influenced by mathematical, personal, managerial, perceptive, and technological concerns not dissimilar from students' instrumental genesis with CAS. As more studies address these complexities, the research community will likely acquire a better understanding of how technology impacts teacher practice in mathematics classrooms.

CONCLUSION

The research surveyed here presents the good and bad that accompany mathematical learning with computer algebra. On the plus side, this technology allows users to look past routine tedium and examine the conceptual underpinnings of the subject. Opportunities to connect important ideas through visual, symbolic, and descriptive channels are at the user's fingertips. Moreover, it is not unusual for CAS to open doors to unconventional topics neglected in standard mathematics curricula—a

“new” mathematics for lack of a better term. Hence, the potential to reshape the fabric of mathematics is very real.

In some reports, it is clear that the richness of classroom discussions and investigations is greatly enhanced for all learners. Computer algebra tools respond with timely feedback—an immediacy that promotes free exploration and discussion of the mathematics on screen. To add, numerous reports convey the nominal effect(s) of CAS on one’s ability to carry out tasks of a drill nature, quelling the concerns of those who fear a negative impact on procedural fluency.

Given the apparent dividends outlined above, research studies have added considerably to answering the question, “How should CAS be used in undergraduate mathematics?” Although a synthesis of the works discussed here leads to inconclusive findings, it remains clear that “black box” usage may do more harm than good. Some of the other uses (e.g., white box, discussion tool) intertwine with pedagogical goals or classroom communication—offering opportunities for deeper learning. Given such a wide swath of advantages (most resonating with current views in mathematics education) the reader may wonder why such powerful tools simply wait in the shadows to be mainstreamed.

The research documents convincingly that any use of CAS comes at a steep price—one that affects both students and teachers. Effective use of computer algebra is not spontaneous and students and teachers alike will need time to become fluent with such tools. The theoretical construct of instrumental genesis has fared well in explaining why this process is so complex for students. Along the trajectory of this genesis, students often struggle with situating the mathematics learned in a CAS environment and mapping

this knowledge to what one should know. Additionally, it is not uncommon for users to find difficulties with syntax or CAS's assortment of unforeseen quirks—most notably in the context of “black box” usage. Finally, some studies report that effective use of CAS rests heavily on one's ability to make a distinction between what is important and less important on the screen. This skill of when to “ignore” certain aspects of CAS is nontrivial and presumably, greatly amplified in the case of the unseasoned user.

Studies of teachers using computer algebra highlight both the challenges of the “new” mathematics and the spontaneity of CAS. Specifically, these studies shed light on the question, “To what degree does CAS influence mathematical content and the manner in which this content is taught?” Quite often, these studies examine the reactions and perspectives of teachers immersed in such situations. First and foremost, computer algebra in mathematics classrooms presents teachers with a different role. As CAS executes algebraic routines with little effort, the technology challenges students to explain why the machine has done what it has done. Teachers are thus repositioned as facilitators of this guidance to mathematical meaning and understanding. As some studies illustrate, this is a foreign place for many teachers and it brings with it a wealth of new challenges and uncertainties. Second, not dissimilar from students' instrumental genesis, teachers are likely to need time and training in order to use CAS effectively. Some studies report that while a teacher may profess to use CAS in certain ways for him/herself, this may not transfer into observable action. Of course, such decisions may be influenced by external and inflexible factors such as an established curriculum, administrative pressures, and/or national testing. The confluence of variables escalates

these challenges, making it, in effect, impossible to identify one factor as the sole agent to change.

Finally, a few comments should be offered with respect to CAS and its timing of implementation. Although the literature casts much light on an “appropriate time” to introduce CAS into the learning experience, it appears that these contributions are greatly influenced by factors such as (a) the nature of the specific topic being learned, (b) the teacher’s conception/comfort level/understanding of this topic, (c) students’ experiences and readiness with CAS, and a host of other variables not yet actualized in the research. Arguments for CAS use “before,” “in conjunction with,” or “after” a mathematical idea is explored can be equally convincing when one examines the specific content under study, the students’ interactions with CAS, and the nature of the learning. It is impractical (perhaps imprudent) to offer a one-size-fits-all retort to such broad questions of tremendous importance. Devoid of a clear and well-defined answer, the literature highlights patches of successes and failures that add to our overall understanding of how CAS fits into the learning experience.

In closing, it seems without question that computer algebra impacts the mathematics that is learned and for some teachers, how the subject is taught. Although the process of becoming a skillful user is often slow to blossom, newer syntax-free interfaces of CAS (e.g., *Maple* and *Mathematica*) may be agents of a sea change. In general, the broad domain of “technology in educational studies” rides the wave of the larger technological revolution. Once an important construct has been identified or a critical finding has emerged, one might find its lifespan cut short by the ever-changing face of new and improved tools. Regardless of the changes we see in the field, one thing

is for certain: CAS has made an entrance, infiltrated the learning experience, and proved to have educational value but not without its share of ills. Just what students and teachers take from these experiences and the degree of their lasting power will be the centerpieces of discussions for years to come.

REFERENCES

- Alpers, B. (2002). CAS as environments for implementing mathematical microworlds. *International Journal of Computer Algebra in Mathematics Education*, 9 (3), 177-204.
- Appel, K., & Haken, W. (1977). The solution of the four color map problem. *Scientific American*, 237 (4), 108-121.
- Arnold, S. (2004). Classroom computer algebra: Some issues and approaches. *Australian Mathematics Teacher*, 60 (2), 17-21.
- Artigue, M. (2000, March). Instrumentation issues and the integration of computer technologies into secondary mathematics teaching. *Proceedings of the annual meeting of the Gesellschaft für Didaktik der Mathematik (GDM)*, 1, 7-17.
- Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7, 245-274.
- Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1011-1049). Charlotte, NC: Information Age Publishing.
- Ayers, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions. *Journal for Research in Mathematics Education*, 19 (3), 246-259.
- Ball, L., Pierce, R., & Stacey, K. (2003, July). *Recognizing equivalent algebraic expressions: An important component of algebraic expectation for working with CAS*. Paper presented at the 27th International Group for the Psychology of Mathematics Education held jointly with the 25th PME-NA Conference, Honolulu, HI.
- Blume, G. (2007). Reflections on John Monaghan's "Computer algebra, instrumentation and the anthropological approach." *International Journal for Technology in Mathematics Education*, 14 (2), 73-76.
- Blume, G.W., & Heid, M.K. (Eds.). (2008). *Research on technology and the teaching and learning of mathematics: Volume 2. Cases and perspectives*. Charlotte, NC: Information Age.
- Bossé, M.J., & Nandakumar, N.R. (2004). Computer algebra systems, pedagogy, and epistemology. *Mathematics and Computer Education*, 38 (3), 298-306.

- Bowers, J., Brandt, J., Stovall, K., & Vargas, M. (2009). Searching for the most amazing thing over the last 20 years: A review of *Research on Technology and the Teaching and Learning of Mathematics, Volume 1 and Volume 2* [Book Review]. *Journal for Research in Mathematics Education*, 40 (5), 564-570.
- Boyce, W.E., & Ecker, J.G. (1995). The computer-oriented calculus course at Rensselaer Polytechnic Institute. *The College Mathematics Journal*, 26 (1), 45-50.
- Bruner, J. (1983). *Child's talk: Learning to use language*. NY: W.W. Norton.
- Buchberger, B. (1989). Should students learn integration rules? *ACM SIGSAM Bulletin*, 24 (1), 10-17.
- Buchberger, B. (2002). Computer algebra: The end of mathematics? *ACM SIGSAM Bulletin*, 36 (1), 3-9.
- Cedillo, T.E. (2001). Towards an algebra acquisition support system: A study based on using graphic calculators in the classroom. *Mathematical Thinking and Learning*, 3 (4), 221-259.
- Cedillo, T., & Kieran, C. (2003). Initiating students into algebra with symbol-manipulating calculators. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 219-239). Reston, VA: National Council of Teachers of Mathematics.
- Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, 19, 221-266.
- Child, J.D. (2002, November). Black box and white box CAS in calculus. *Proceedings of the Annual International Conference on Technology in Collegiate Mathematics*, 1, 44-48.
- de Alwis, T. (2002, November). Computer algebra systems in mathematics education—computation and visualization. *Proceedings of the Annual International Conference on Technology in Collegiate Mathematics*, 1, 16-21.
- Dick, T.P. (2008). Keeping the faith: Fidelity in technological tools for mathematics education. In G.W. Blume & M.K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 2. Cases and perspectives* (pp. 333-339). Charlotte, NC: Information Age.
- Dick, T.P. (2008). Personal e-mail communication.
- Drijvers, P. (2000). Students encountering obstacles using a CAS. *International Journal*

- of Computers for Mathematical Learning*, 5, 189-209.
- Drijvers, P. (2002). Learning mathematics in a computer algebra environment: Obstacles are opportunities. *Zentralblatt fuer Didaktik der Mathematik*, 34 (5), 221-228.
- Drijvers, P. (2003). Algebra on screen, on paper, and in the mind. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 241-267). Reston, VA: National Council of Teachers of Mathematics.
- Drijvers, P., & Trouche, L. (2008). From artifacts to instruments: A theoretical framework behind the orchestra metaphor. In G.W. Blume & M.K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 2. Cases and perspectives* (pp. 363-391). Charlotte, NC: Information Age.
- Drijvers, P., & van Herwaarden, O. (2000). Instrumentation of ICT-tools: The case of algebra in a computer algebra environment. *International Journal of Computer Algebra in Mathematics Education*, 7 (4), 255-275.
- Edwards, L. (1998). Embodying mathematics and science: Microworlds as representations. *Journal of Mathematical Behavior*, 17 (1), 53-78.
- Fey, J.T., Cuoco, A., Kieran, C., McMullin, L., & Zbiek, R.M. (Eds.). (2003). *Computer algebra systems in secondary school mathematics education*. Reston, VA: National Council of Teachers of Mathematics.
- Garofalo, J. (1989). Beliefs and their influence on mathematical performance. *Mathematics Teacher*, 82, 502-505.
- Goldenberg, E.P. (1988). Mathematics, metaphors, and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *Journal of Mathematical Behavior*, 7, 135-173.
- Goldenberg, E.P. (2003). Algebra and computer algebra. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 9-30). Reston, VA: National Council of Teachers of Mathematics.
- Gordon, S.P., & Kleinstein, A.H. (2002). The implications of hand-held computer algebra system calculators throughout the mathematics curriculum. *The AMATYC Review*, 23 (2), 61-73.
- Guin, D., & Delgoulet, J. (1996). *Etude des modes d'appropriation de calculatrices graphiques et symboliques dans une classe de seconde*. IREM de Montpellier.
- Guin, D., & Trouche, L. (1999). The complex process of converting tools into

- mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning*, 3, 195-227.
- Guin, D., & Trouche, L. (2002). Mastering by the teacher of the instrumental genesis in CAS environments: Necessity of instrumental orchestrations. *Zentralblatt fuer Didaktik der Mathematik*, 34 (5), 204-211.
- Heid, M.K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education*, 19 (2), 3-25.
- Heid, M.K. (1997). The technological revolution and the reform of school mathematics. *American Journal of Education*, 106 (1), 5-61.
- Heid, M.K. (2003). Theories for thinking about the use of CAS in teaching and learning mathematics. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 33-52). Reston, VA: National Council of Teachers of Mathematics.
- Heid, M.K., & Blume, G.W. (2008). Technology and the teaching and learning of mathematics: Cross-content implications. In M.K. Heid & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 1. Research syntheses* (pp. 419-431). Charlotte, NC: Information Age.
- Heid, M.K., & Blume, G.W. (Eds.). (2008). *Research on technology and the teaching and learning of mathematics: Volume 1. Research syntheses*. Charlotte, NC: Information Age.
- Heid, M.K., Blume, G.W., Hollebrands, K.F., & Piez, C. (2002). Computer algebra systems in mathematics instruction: Implications from research. *Mathematics Teacher*, 95 (8), 586-591.
- Heid, M.K., & Edwards, M.T. (2001). Computer algebra systems: Revolution or retrofit for today's mathematics classrooms? *Theory into Practice*, 40 (2), 128-136.
- Heid, M.K., Hollebrands, K.F., & Iseri, L.W. (2002). Reasoning and justification with examples from technological environments. *Mathematics Teacher*, 95 (3), 210-216.
- Hillel, J., Lee, L., Laborde, C., & Linchevski, L. (1992). Basic functions through the lens of computer algebra systems. *Journal of Mathematical Behavior*, 11, 119-158.
- Hoyles, C., & Noss, R. (1992). A pedagogy for mathematical microworlds. *Educational Studies in Mathematics*, 23 (1), 31-57.
- Hoyles, C., Noss, R., & Kent, P. (2004). On the integration of digital technologies into

- mathematics classrooms. *International Journal of Computers for Mathematical Learning*, 9, 309-326.
- Judson, P.T. (1990). Elementary business calculus with computer algebra. *Journal of Mathematical Behavior*, 9, 153-157.
- Kaput, J. (1992). Technology and mathematics education. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). New York: Macmillan.
- Kaput, J. (1994). Democratizing access to calculus: New routes to old routes. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77-156). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Kendal, M., & Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. *International Journal of Computers for Mathematical Learning*, 6, 143-165.
- Kieran, C., & Drijvers, P. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning*, 11, 205-263.
- Kilpatrick, J., & Davis, R.B. (1993). Computers and curriculum change in mathematics. In C. Keitel & K. Ruthven (Eds.), *Learning from computers: Mathematics education and technology* (pp. 203-221). Berlin: Springer.
- Kutzler, B. (2003). CAS as pedagogical tools for teaching and learning mathematics. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 53-71). Reston, VA: National Council of Teachers of Mathematics.
- Lagrange, J.-B. (1996). Analyzing actual use of a computer algebra system in the teaching and learning of mathematics. *International DERIVE Journal*, 3 (3), 91-108.
- Lagrange, J.-B. (1999a). Complex calculators in the classroom: Theoretical and practical reflections on teaching pre-calculus. *International Journal of Computers for Mathematical Learning*, 4, 51-81.
- Lagrange, J.-B. (1999b). Techniques and concepts in pre-calculus using CAS: A two year classroom experiment with the TI-92. *International Journal of Computer Algebra in Mathematics Education*, 6 (2), 143-165.
- Lagrange, J.-B. (2003). Learning techniques and concepts using CAS: A practical and theoretical reflection. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M.

- Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 269-283). Reston, VA: National Council of Teachers of Mathematics.
- Lester, F.K. (2007). Personal e-mail communication.
- Lumb, S., Monaghan, J., & Mulligan, S. (2000). Issues arising when teachers make extensive use of computer algebra. *The International Journal of Computer Algebra in Mathematics Education*, 7 (4), 223-240.
- McCallum, W.G. (2003). Thinking out of the box. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 73-86). Reston, VA: National Council of Teachers of Mathematics.
- Monaghan, J. (2005, October). *Computer algebra, instrumentation and the anthropological approach*. Paper presented at the Fourth Computer Algebra in Mathematics Education Symposium, Roanoke, VA. Retrieved on December 12, 2009 from <http://www.lkl.ac.uk/research/came/events/CAME4/CAME4-topic1-Monaghan-paper.pdf>.
- Monaghan, J., & Ozmentar, M.F. (2006). Comments on the co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection. *International Journal of Computers for Mathematical Learning*, 11, 351-360.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht, The Netherlands: Kluwer Academic.
- O'Callaghan, B. (1998). Computer-intensive algebra and students' conceptual knowledge of functions. *Journal for Research in Mathematics Education*, 29 (1), 21-40.
- Palmiter, J.R. (1991). Effects of computer algebra systems on concept and skill acquisition in calculus. *Journal for Research in Mathematics Education*, 22 (2), 151-156.
- Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books.
- Paulos, J.A. (1988). *Innumeracy*. New York: Hill & Wang.
- Pea, R.D. (1987). Cognitive technologies for mathematics education. In A.H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 89-122). Hillsdale, NJ: Erlbaum.

- Piaget, J. (1970). *Genetic epistemology*. New York: W.W. Norton.
- Piaget, J. (1976). *The grasp of consciousness*. Cambridge, MA: Harvard University Press (originally published 1974).
- Piaget, J. (1978). *Success and understanding*. Cambridge, MA: Harvard University Press (originally published 1974).
- Pierce, R., & Ball, L. (2009). Perceptions that may affect teachers' intention to use technology in secondary mathematics classes. *Educational Studies in Mathematics*, 71, 299-317.
- Pierce, R., & Stacey, K. (2001). Observations on students' responses to learning in a CAS environment. *Mathematics Education Research Journal*, 13 (1), 28-46.
- Pierce, R., & Stacey, K. (2002). Algebraic insight: The algebra needed to use computer algebra systems. *Mathematics Teacher*, 95 (8), 622-627.
- Pierce, R., & Stacey, K. (2004a). A framework for monitoring progress and planning teaching towards the effective use of computer algebra systems. *International Journal of Computers for Mathematical Learning*, 9, 59-93.
- Pierce, R., & Stacey, K. (2004b, July). Learning to use CAS: Voices from a classroom. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, 4, 25-32.
- Ruthven, K. (2002). Instrumenting mathematical activity: Reflections on key studies of the educational use of computer algebra systems. *International Journal of Computers for Mathematical Learning*, 7, 275-291.
- Schneider, E. (2000). Teacher experiences with the use of a CAS in a mathematics classroom. *International Journal of Computer Algebra in Mathematics Education*, 7 (2), 119-141.
- Schoenfeld, A.H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. *Educational Psychologist*, 23 (2), 145-166.
- Schoenfeld, A.H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.
- Schultz, J.E. (2003). To CAS or not to CAS? In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.), *Computer algebra systems in secondary school*

- mathematics education* (pp. 163-171). Reston, VA: National Council of Teachers of Mathematics.
- Texas Instruments. (2009). *TI-92. Voyage 200. TI-Nspire*. Dallas, TX: Texas Instruments, Inc.
- Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, 9 (3), 281-307.
- Vérillon, P. (2000). Revisiting Piaget and Vygotsky: In search of a learning model for technology education. *Journal of Technology Studies*, 26 (1), 1-9.
- Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought[t] in relation to instrumented activity. *European Journal of Psychology in Education*, 9 (3), 77-101.
- Vygotsky, L.S. (1930/1985). La méthode instrumentale en psychologie [The instrumental method in psychology]. In B. Schneuwly & J.P. Bronckart (Eds.), *Vygotsky aujourd'hui* (pp. 39-47). Neûchatel: Delachaux et Niestlé.
- Waterloo Maple. (2009). *Maple* (Version 13). Waterloo, Ontario, Canada: Waterloo Maple Software, Inc.
- Wolfram Research. (2009). *Mathematica* (Version 7). Champaign, IL: Wolfram Research, Inc.
- Wolfram, S. (2002). *A new kind of science*. Champaign, IL: Wolfram Media.
- Zbiek, R.M. (1995, October). *Her math, their math: An in-service teacher's growing understanding of mathematics and technology and her secondary students' algebra experience*. Paper presented at the annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education, Columbus, OH. (ERIC Document Reproduction Service No. ED389610)
- Zbiek, R.M. (2002). Influences on mathematics teachers' transitional journeys in teaching with CAS. *International Journal of Computer Algebra in Mathematics Education*, 9 (2), 129-137.
- Zbiek, R.M. (2003). Using research to influence teaching and learning with computer algebra systems. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 197-216). Reston, VA: National Council of Teachers of Mathematics.
- Zbiek, R.M., & Heid, M.K. (2009). Using computer algebra systems to develop big ideas

- in mathematics. *Mathematics Teacher*, 102 (7), 540-544.
- Zbiek, R.M., Heid, M.K., Blume, G.W., & Dick, T.P. (2007). Research on technology in mathematics education: A perspective of constructs. In F.K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 1169-1207). Charlotte, NC: Information Age Publishing.
- Zbiek, R.M., & Hollebrands, K. (2008). A research-informed view of the process of incorporating mathematics technology into classroom practice by in-service and prospective teachers. In M.K. Heid & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 1. Research syntheses* (pp. 287-344). Charlotte, NC: Information Age.