MATH 256 **Name:** Key
Exam 1
10/3/16

DIRECTIONS: Calculators are permitted this exam. However, answers based solely on calculator results are not acceptable (unless it says otherwise). You must show all work to receive full credit. Good luck.

1. **(20 points)** Define $A=\begin{bmatrix}6 & -1\\ 0 & 4\\ 2 & -3\end{bmatrix}$, $B=\begin{bmatrix}1 & 4\\ -1 & 5\\ 1 & 7\end{bmatrix}$, and $C=\begin{bmatrix}-3 & -2\\ 4 & 2\end{bmatrix}$. Find the following, if possible. Otherwise, say "undefined."

(a) $2A-B$

$$\begin{bmatrix}12 & -2\\ 0 & 8\\ 4 & -6\end{bmatrix}-\begin{bmatrix}1 & 4\\ -1 & 5\\ 1 & 7\end{bmatrix}=\boxed{\begin{bmatrix}11 & -6\\ 1 & 3\\ 3 & -13\end{bmatrix}}$$

(b) BC

$$\begin{bmatrix}1 & 4\\ -1 & 5\\ 1 & 7\end{bmatrix}\begin{bmatrix}-3 & -2\\ 4 & 2\end{bmatrix}=\boxed{\begin{bmatrix}13 & 6\\ 23 & 12\\ 25 & 12\end{bmatrix}}$$

(c) $(BC)^{-1}$

BC above is not square;
$(BC)^{-1}$ does not exist / undefined

(d) CA^T

$$\begin{bmatrix}-3 & -2\\ 4 & 2\end{bmatrix}\begin{bmatrix}6 & 0 & 2\\ -1 & 4 & -3\end{bmatrix}=\boxed{\begin{bmatrix}-16 & -8 & 0\\ 22 & 8 & 2\end{bmatrix}}$$

so no matrix A could possibly exist.

2. **(10 points)** Explain why there is no solution A to the matrix equation $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

let $B=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ so $BA=I_2$. This implies that A is an inverse of B. However B has no inverse since it is not row equivalent to I_2. Note: $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$

3. **(12 points)** Solve the following system, if possible. Show your work.

$$\begin{aligned} x_1 \quad\quad -3x_3 &= -2 \\ 3x_1 + x_2 - 2x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right] \quad \begin{array}{l} R_1(-3) + R_2 \to R_2 \\ R_1(-2) + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right] \quad R_2(-2) + R_3 \to R_3 \qquad \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{array}{l} R_3(3) + R_1 \to R_1 \\ R_3(-7) + R_2 \to R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\boxed{(x_1, x_2, x_3) = (4, -3, 2)}$$

4. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$.

(a) **(7 points)** Compute A^{-1}.

$$A^{-1} = \frac{1}{(1)(7)-(2)(4)} \begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}}$$

(b) **(6 points)** Use the result in part (a) to solve the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 1/2 \\ -3 \end{bmatrix}$.

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1/2 \\ -3 \end{bmatrix} = \boxed{\begin{bmatrix} -19/2 \\ 5 \end{bmatrix}}$$

(c) **(6 points)** Show clearly that **b** can be written as a linear combination of the columns of A.

$$\begin{bmatrix} 1/2 \\ -3 \end{bmatrix} = -\frac{19}{2} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

5. **(12 points)** Calculate the determinant of $A=\begin{bmatrix}1 & 4 & 1\\ 2 & -1 & 0\\ 0 & 18 & 4\end{bmatrix}$ by using a cofactor expansion.

Col 1 expansion:

$$\det A = 1(-1)^{1+1}\begin{vmatrix}-1 & 0\\ 18 & 4\end{vmatrix} + 2(-1)^{2+1}\begin{vmatrix}4 & 1\\ 18 & 4\end{vmatrix} + 0(-1)^{3+1}\begin{vmatrix}4 & 1\\ -1 & 0\end{vmatrix}$$

$$= (1)(-4) - 2(16-18) + 0$$

$$= \boxed{0}$$

6. **(12 points)** Choose **one** of the following problems below.

(a) Prove that if $A^2 = A$, then $(I-2A)=(I-2A)^{-1}$.

Show that $I-2A$ is its own inverse

(b) Let A and B be matrices such that the product AB is defined. Prove that if A has two identical rows, then the corresponding two rows of AB are also identical.

(A) Consider

$$(I-2A)(I-2A) = I^2 - I(2A) - 2AI + 4A^2$$
$$= I - 2A - 2A + 4A^2$$
$$= I - 4A + 4A^2$$
$$= I - 4A + 4A \leftarrow$$

since $A^2 = A$

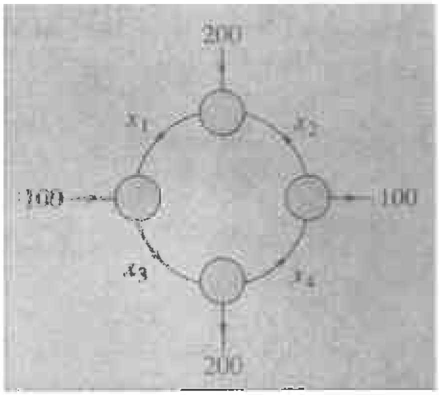
$$= I + 0$$
$$= I$$

This proves that the inverse of $I-2A$ is itself.

That is,

$$(I-2A)^{-1} = I - 2A$$

7. **(15 points)** Solve the network flow problem as seen in the diagram below. Find each x_i, $i = 1,2,3$ and 4.



Source: Larson (2013)

North : $x_2 + 200 = x_1$

East : $x_4 = 100 + x_2$

South : $x_3 = x_4 + 200$

West : $100 + x_1 = x_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 200 \\ 0 & -1 & 0 & 1 & 100 \\ 0 & 0 & 1 & -1 & 200 \\ -1 & 0 & 1 & 0 & 100 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 200 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 200 \\ 0 & -1 & 1 & 0 & 300 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 200 \\ 0 & 0 & 1 & -1 & 200 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & -1 & -100 \\ 0 & 0 & 1 & -1 & 200 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_4 = 100$

$x_2 - x_4 = -100$

$x_3 - x_4 = 200$

let $x_4 = t$

t is real

$x_1 = 100 + t$

$x_2 = -100 + t$

$x_3 = 200 + t$

$$\boxed{\{(100+t, -100+t, 200+t, t) \mid t \text{ is real}\}}$$