MATH 166 Name: Key

Test 2 / Summer 2017

DIRECTIONS: This is a closed book, closed notes exam. Calculators are permitted but answers based solely on calculator results are unacceptable. You must still show all work to receive full credit. Good luck.

1. **(15 points)** Find the derivative of each function. Simplify to a reasonable point (eliminate complex fractions, combine like terms, etc.).

(a) $f(x) = \sin^4 3x$ $= (\sin 3x)^4$

$$f'(x) = 4(\sin 3x)^3 \cos 3x \cdot 3$$

$$= \boxed{12 \sin^3 3x \cos 3x}$$

(b) $k(\theta) = \theta^2\sqrt{2-\theta^2}$ $= \theta^2(2-\theta^2)^{1/2}$

$$k'(\theta) = \theta^2 \tfrac{1}{\cancel{2}}(2-\theta^2)^{-1/2}(-\cancel{2}\theta) + \sqrt{2-\theta^2}\cdot(2\theta)$$

$$= -\frac{\theta^3}{\sqrt{2-\theta^2}} + 2\theta\sqrt{2-\theta^2}\cdot\frac{\sqrt{2-\theta^2}}{\sqrt{2-\theta^2}}$$

$$= -\frac{\theta^3}{\sqrt{2-\theta^2}} + \frac{2\theta(2-\theta^2)}{\sqrt{2-\theta^2}} = \boxed{\frac{4\theta - 3\theta^3}{\sqrt{2-\theta^2}}}$$

(c) $y = \sqrt{\dfrac{t}{t+3}}$ $= \left(\dfrac{t}{t+3}\right)^{1/2}$

$$\frac{dy}{dt} = \frac{1}{2}\left(\frac{t}{t+3}\right)^{-1/2} \frac{(t+3)\cdot 1 - t\cdot 1}{(t+3)^2}$$

$$= \frac{1}{2}\left(\frac{t+3}{t}\right)^{1/2}\frac{3}{(t+3)^2}$$

$$= \boxed{\frac{3}{2\sqrt{t}\,(t+3)^{3/2}}}$$

2. **(9 points)** Find the equation of the tangent line to $x^2 - xy - y^2 = 1$ at the point $(2,1)$ (the graph is a hyperbola).

$$\frac{d}{dx}(x^2 - xy - y^2) = \frac{d}{dx}(1)$$

$$2x - (xy' + y \cdot 1) - 2yy' = 0$$

$$2x - y = xy' + 2yy'$$

$$2x - y = y'(x + 2y)$$

$$y' = \frac{2x - y}{x + 2y} \quad \text{(slope)}$$

$$y'(2,1) = \frac{4-1}{2+2} = \frac{3}{4}$$

$$y = mx + b$$

$$1 = \frac{3}{4}(2) + b$$

$$b = -1/2$$

$$\boxed{y = \frac{3}{4}x - \frac{1}{2}}$$

3. At time $t = 0$, a diver jumps from a diving board that is 32 feet above the water. The position of the diver is given by $s(t) = -16t^2 + 16t + 32$, where s is measured in feet and t is measured in seconds.

(a) **(4 points)** When does the diver hit the water?

$s = 32$ (initially) When is $s = 0$?

$$0 = -16t^2 + 16t + 32$$

$$= -16(t^2 - t - 2)$$

$$0 = -16(t-2)(t+1)$$

$$\boxed{t = 2 \text{ sec}}$$ (only relevant solution)

(b) **(4 points)** What is the diver's velocity at impact?

$$v(t) = s'(t)$$

$$= -32t + 16$$

$$v(2) = -64 + 16$$

$$= \boxed{-48 \text{ ft/sec}}$$

4. **(6 points)** Locate the absolute extrema of the function $y = 3x^{2/3} - 2x$ on the interval $[-1, 1]$.

$$\frac{dy}{dx} = \cancel{3}\,\frac{2}{\cancel{3}}x^{-1/3} - 2$$

$$= \frac{2}{x^{1/3}} - 2 \cdot \frac{x^{1/3}}{x^{1/3}}$$

$$= \frac{2 - 2x^{1/3}}{x^{1/3}}$$

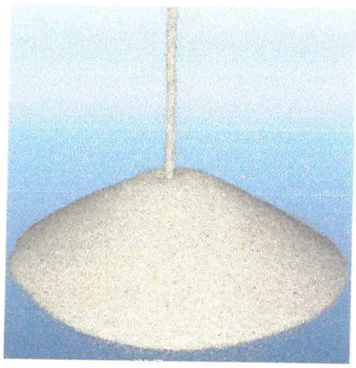
undefined at $x = 0$

$x = 1$ makes $\frac{dy}{dx} = 0$

$\boxed{x = 0, 1}$ ← Test values

x	$y = 3x^{2/3} - 2x$	
0	0	← minimum
1	1	
-1	5	← maximum

5. **(8 points)** Sand falls from an overhead bin, accumulating in a conical pile with a radius that is always three times its height. If the sand falls from the bin at a rate of 120 ft^3/min, how fast is the height of the sand pile changing when the pile is 10 ft high? Note: $V = \frac{1}{3}\pi r^2 h$



Given: $r = 3h$ $\left(\text{so } \frac{dr}{dt} = 3\frac{dh}{dt}\right)$

$\frac{dV}{dt} = 120 \text{ ft}^3/\text{min}$, $\frac{dh}{dt} = ??$ when

$h = 10$ ft (so $r = 30$ ft)

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[r^2\frac{dh}{dt} + h(2r)\frac{dr}{dt}\right]$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left[r^2\frac{dh}{dt} + 2hr \cdot 3\frac{dh}{dt}\right]$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2 + 6hr\right)\frac{dh}{dt}$$

$$\frac{dV}{dt} \Big/ \frac{1}{3}\pi(r^2 + 6hr) = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{120}{\pi/3\left(30^2 + 6(10)(30)\right)}$$

$$\approx \boxed{0.042 \text{ ft/min}}$$

6. A machine is causing a particle to move along the x-axis so that its position at time t is given by $x(t) = (t-4)^2$, where t is in seconds.

(a) **(3 points)** What is the particle's velocity at $t = 2$?

$$v(t) = x'(t) = 2(t-4)$$

$$v(2) = 2(-2) = \boxed{-4 \text{ units/sec}}$$

(b) **(4 points)** The machine stops suddenly at $t = 3$ releasing the particle. As the particle continues, where will it be 5 seconds after the machine stops? Explain your thinking.

$x(3) = 1$ unit

$v(3) = -2$ units/sec

data at moment of breakdown

$x(t) = 1 - 2(5) = \boxed{-9}$ ← future location

initial position

velocity

elapsed time

7. Consider the function $f(x)=\sqrt{x+3}$.

$(1, f(1)) = (1, 2)$ (point)

(a) **(5 points)** Find the linear approximation $L(x)$ to $f(x)=\sqrt{x+3}$ near $a=1$.

$$f'(x) = \frac{1}{2}(x+3)^{-1/2} \cdot 1$$

$$= \frac{1}{2\sqrt{x+3}}$$

$$f'(1) = \frac{1}{2\sqrt{4}} = \frac{1}{4} \text{ (slope)}$$

$$y = mx + b$$

$$2 = \frac{1}{4} \cdot 1 + b \Rightarrow b = \frac{7}{4}$$

$$\boxed{y = \frac{1}{4}x + \frac{7}{4} = L(x)}$$

(b) **(5 points)** Explain and show how you could use part (a) to get a reliable estimate for $\sqrt{3.98}$.

$$f(0.98) = \sqrt{0.98+3} = \sqrt{3.98} \approx L(0.98)$$

because $L(x)$ estimates $f(x)$ near $x=1$ (here, $x = 0.98$)

$$L(0.98) = \frac{1}{4}(0.98) + \frac{7}{4}$$

$$= 1.995 \text{ (approximation for } \sqrt{3.98}\text{)}$$

A calculator gives $\sqrt{3.98} \approx 1.99499$

(indeed, L does a very good job)