MATH 256
Exam 2
November 14, 2016

Name: Key

DIRECTIONS: Calculators are permitted this exam. However, answers based solely on calculator results are not acceptable (unless it says otherwise). You must show all work to receive full credit. Good luck.

1. Suppose A is 4×4 with $\det A = -1$. Determine the values of each of the below.

(a) **(5 points)** $\det A^T = \det A = -1$

(b) **(5 points)** $\det A^{100} = (\det A)^{100} = (-1)^{100} = 1$

(c) **(5 points)** $\det(3A) = 3^4(\det A) = 81(-1) = -81$

$\uparrow$ A is 4×4

2. Disprove each of the following statements.

(a) **(7 points)** The set of all 2×2 invertible matrices with the standard operations is a vector space.

$\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ must be in the vector space

yet $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible. Set does not contain the zero vector.

(b) **(7 points)** The set $\{(x,y) \mid x \ge 0, y \text{ is a real number}\}$ with the standard operations in $\mathbb{R}^2$ is a vector space.

Consider $(1,4)$ in the set.

$-2(1,4) = (-2,-8)$ is no longer in the set since $-2 \not\geq 0$. Not closed under scalar multiplication.

3. **(8 points)** Consider the subset of $\mathbb{R}^3$ given by $W=\{(x_1, x_2, 4) \mid x_1, x_2 \in \mathbb{R}\}$. Is W a subspace of $\mathbb{R}^3$? Why or why not?

No, W is not a subspace of R^3.

Take $(x_1, x_2, 4), (y_1, y_2, 4) \in W$

$$(x_1, x_2, 4) + (y_1, y_2, 4) = (x_1+y_1, x_2+y_2, 8) \notin W$$

Not closed under vector addition

4. Let $p(t)=5t^2+6t+5 \in P_2$, where P_2 is the vector space of polynomials of degree two or less with the standard operations.

(a) **(7 points)** Consider the set $S=\{\overbrace{t^2+2t+1}^{\vec{v}_1},\overbrace{t^2-2t+1}^{\vec{v}_2},\overbrace{3t}^{\vec{v}_3}\}$. Can $p(t)$ be written as a linear combination of the functions in S? Show detailed reasoning.

$$c_1(t^2+2t+1) + c_2(t^2-2t+1) + c_3(3t) = 5t^2+6t+5$$

Equate coefficients $\Rightarrow$

$$\begin{aligned} c_1 + c_2 \quad &= 5 \\ 2c_1 - 2c_2 + 3c_3 &= 6 \\ c_1 + c_2 \quad &= 5 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 2 & -2 & 3 & 6 \\ 1 & 1 & 0 & 5 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc} 1 & 0 & 3/4 & 4 \\ 0 & 1 & -3/4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↑ Does this have a solution?

Ex:

$\vec{p} = \vec{v}_1 + 4\vec{v}_2 + 4\vec{v}_3$

← Has infinitely many solutions so answer is yes.

(b) **(7 points)** Does S (from part (a)) span P_2? Show detailed reasoning.

Similar to (A) but asking, "can any $p(t) = at^2 + bt + c$ be generated?" Leads to

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 2 & -2 & 3 & b \\ 1 & 1 & 0 & c \end{array}\right] \leftarrow \text{Well, not consistent unless } a = c$$

$\Rightarrow$ Not always solvable.

$\Rightarrow$ S does not span P_2.

NO

5. **(8 points)** Suppose that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set in a vector space V. Prove or disprove: $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}$ is a linearly independent set in V.

By assumption, $c_1\vec{u} + c_2\vec{v} = \vec{0}$ implies $c_1 = c_2 = 0$.

What about $d_1(\vec{u}+\vec{v}) + d_2(\vec{u}-\vec{v}) = \vec{0}$?

Must $d_1 = d_2 = 0$? $(*)$

$$d_1(\vec{u}+\vec{v}) + d_2(\vec{u}-\vec{v}) = \vec{0}$$

$$\Rightarrow (d_1+d_2)\vec{u} + (d_1-d_2)\vec{v} = \vec{0}$$

Since $\{\vec{u},\vec{v}\}$ are LI, $d_1 + d_2 = 0$ & $d_1 - d_2 = 0$

Examine $\begin{cases} d_1 + d_2 = 0 \\ d_1 - d_2 = 0 \end{cases} \rightarrow \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \neq 0 \Rightarrow$ only trivial sol'n $d_1 = d_2 = 0$ see $(*)$

$\Rightarrow \{\vec{u}+\vec{v}, \vec{u}-\vec{v}\}$ $\underline{\underline{LI}}$.

6. Only one of the sets below is a basis for $\mathbb{R}^3$.

$$S_1 = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \right\}, \quad S_2 = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ -9 \end{bmatrix} \right\}, \quad S_3 = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Clearly indicate

(a) **(6 points)** which set is a basis for $\mathbb{R}^3$,

S_2

(b) **(6 points)** the reason for your answer, and

S_1 does not span $\mathbb{R}^3$ (see entry position 3)

S_3 is linearly dependent (4 vectors in $\mathbb{R}^3$)

(c) **(6 points)** the reasons that the other two sets are not bases for $\mathbb{R}^3$.

$$\begin{vmatrix} 2 & 1 & 7 \\ 0 & 5 & 8 \\ 0 & 0 & -9 \end{vmatrix} = -90 \neq 0$$

7. **(8 points)** State any four equivalent statements to the statement "A is an invertible $n \times n$ matrix." **Be precise**. For each of the four statements you provide, explain why the equivalence is apparent. Proofs are not necessary here; explanations will suffice.

A is invertible

$A\vec{x} = \vec{b}$ has a unique solution

$\vec{x} = A^{-1}\vec{b}$ (1)

(6) $A\vec{x} = \vec{0}$ has only trivial solution, etc....

Directly related: $\text{Col}\,A = \mathbb{R}^n$ (2)

(columns of A generate all of $\mathbb{R}^n$)

(3) $\dim(\text{Col}\,A) = n \Rightarrow$ (4) $\text{rank}\,A = n \Rightarrow$ (5) $\text{Nul}\,A = \{\vec{0}\}$

8. Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

let $x_5 = t$, $x_4 = s$, $x_2 = r$

$x_1 = 2r + s - 3t$

$x_3 = -2s + 2t$

(a) **(8 points)** Find a basis for the null space of A (use your calculator).

$$[A \mid \vec{0}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \end{aligned}\right\}$$

$$\vec{x} = \begin{bmatrix} 2r + s - 3t \\ r \\ -2s + 2t \\ s \\ t \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

(b) **(7 points)** List any vector in the column space of A.

e.g., $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \in \text{Col}\,A$

Basis for NulA:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$