MATH 166
Test 3 / Summer 2017

Name: Key

DIRECTIONS: This is a closed book, closed notes exam. Calculators are permitted but answers based solely on calculator results are unacceptable. You must still show all work to receive full credit. Good luck.

1. **(15 points)** Give a complete, full analysis of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 12$$

Include intervals of increase/decrease, extrema, concavity, and inflection points (as well as anything else relevant to the graph). Sketch the graph neatly by using all of the information found. **Clearly label your work for full credit.**

$(0,12) \leftarrow$ intercept

$f(x) = 2x^3 - 3x^2 - 12x + 12$

$f'(x) = 6x^2 - 6x - 12 = 0$ ↑ set

$6(x^2 - x - 2) = 0$

$6(x-2)(x+1) = 0$

$x = -1, 2$ ← critical #s

$+ \quad - \quad +$ sign of f'

$-1 \quad 2$

$$\boxed{\begin{array}{l} \text{INC}: (-\infty,-1) \cup (2,\infty) \\ \text{DEC}: (-1,2) \\ \text{Min}: (2, f(2)) = (2,-8) \\ \text{Max}: (-1, f(-1)) = (-1, 19) \end{array}}$$

$f''(x) = 12x - 6 = 0$ ↙ set

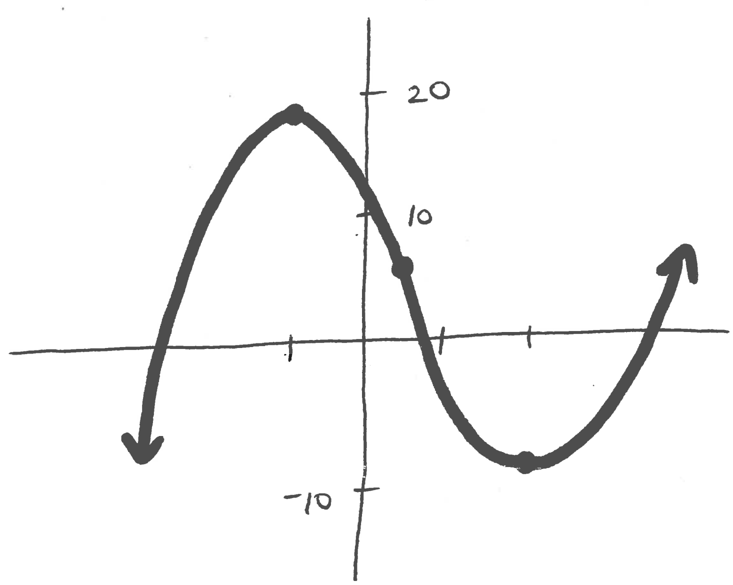
$x = \frac{1}{2}$ ppoi

$- \quad +$ sign of f''

$\frac{1}{2}$

$$\boxed{\begin{array}{l} \text{CC}\downarrow: (-\infty, 1/2) \\ \text{CC}\uparrow: (1/2, \infty) \end{array}}$$

$$\boxed{\text{inflection point}: \left(\tfrac{1}{2}, f(\tfrac{1}{2})\right) = \left(\tfrac{1}{2}, \tfrac{11}{2}\right)}$$



2. **(10 points)** The Jamaican sprinter Usain Bolt set a world record of 9.58 seconds in the 100 meter dash in the summer of 2009. Did his speed ever exceed 37 km/hr during the race? How do you know? Give detailed reasoning.

$$\text{Average speed} = \frac{100\,m}{9.58\,sec} \approx 10.44\ m/s$$

$$10.44\ \frac{m}{s} \times \frac{3600\,s}{1\,hr} \times \frac{1\,Km}{1000\,m} \approx \boxed{37.58\ Km/hr}$$

Average speed

MVT guarantees that Bolt was traveling at precisely this speed (37.58 km/hr) at some point during the race $\Rightarrow$ Ans: Yes

3. **(8 points)** Choose **one** of the problems below to submit. Place a checkmark in the box next to the problem you are doing.

☐ Problem A: Determine where the function $y = x\sqrt{6-x}$ is increasing/decreasing. Use this to determine any extreme values of the function.

☐ Problem B: Consider the function $g(x) = \frac{x}{x-3}$. Determine the intervals where the graph is concave upward/downward and identify any points of inflection.

Problem A

$y = x\sqrt{6-x}$ Note: $x \le 6$

$$\frac{dy}{dx} = x\,\tfrac{1}{2}(6-x)^{-1/2}(-1) + \sqrt{6-x}$$

$$= \frac{-x}{2\sqrt{6-x}} + \sqrt{6-x}\,\frac{2\sqrt{6-x}}{2\sqrt{6-x}}$$

$$= \frac{-x + 2(6-x)}{2\sqrt{6-x}}$$

$$= \boxed{\frac{12-3x}{2\sqrt{6-x}}}$$

$x = 4$ (critical value)

++++ ‑‑‑‑‑ sign of y'

4 6

Inc: $(-\infty,4)$ Decr: $(4,6)$ Max: $(4, 4\sqrt{2})$

Problem B

$$g(x) = \frac{x}{x-3}$$

$$g'(x) = \frac{(x-3)\cdot 1 - x\cdot 1}{(x-3)^2} = \frac{-3}{(x-3)^2}$$

$$g''(x) = -3(-2)(x-3)^{-3}\cdot 1$$

$$= \frac{6}{(x-3)^3} \quad \text{test } x=3$$

‑‑‑‑‑ ++++ sign of g''

3

CC ↓: $(-\infty, 3)$

CC ↑: $(3, \infty)$

No inflection pts

VA @ $x = 3$

4. **(8 points)** Prove that $\lim_{x\to\infty}\frac{\sin x}{x}=0$ (Use the Squeeze Theorem).

$-1 \le \sin \le 1$ (known from Trigonometry)

since $x \to \infty$, $x > 0$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x} \to 0$$

0

Now let $x \to \infty$

Thus $\frac{\sin x}{x} \to 0$ as $x \to \infty$ (Squeeze Thm)

5. **(8 points)** Use Newton's method $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$ to approximate a real zero of $f(x)=1-2x-x^3$ to three decimal places. Use a grapher to make an initial guess.

$f'(x) = -2-3x^2$

$x_1 = 0$ to start

$$x_{n+1} = x_n - \frac{1-2x_n-x_n^3}{-2-3x_n^2}$$

n	x_n
1	0
2	0.5
3	0.4545
4	0.453 *
5	0.453

$\boxed{x \approx 0.453}$

6. **(10 points)** Determine the function $f(x)$ given that $f'(x)=\frac{x^2+\sqrt{x}}{x}$ and $f(1)=3$.

$$f'(x) = \frac{x^2}{x} + \frac{\sqrt{x}}{x} = x + x^{-1/2}$$

then $f(x) = \frac{x^2}{2} + \frac{x^{1/2}}{1/2} + C$

$$\Rightarrow f(x) = \frac{x^2}{2} + 2\sqrt{x} + C$$

$$f(1) = 3 = \frac{1}{2} + 2(1) + C \Rightarrow C = \frac{1}{2}$$

$$\boxed{f(x) = \frac{x^2}{2} + 2\sqrt{x} + \frac{1}{2}}$$

7. **(8 points)** Suppose a stone is dropped from a cliff. The stone hits the water at 120 feet/sec. What is the height of the cliff? Use acceleration $= -32$ feet / sec^2.

$a = -32 \text{ ft}/s^2$

$v(t) = -32t + c$

Since $v(0) = 0$,

(stone is dropped)

$c = 0$

$\Rightarrow v(t) = -32t$

Then $s(t) = -16t^2 + d$

$s(0) = d$

↑

height of cliff

$v_f = -120$ ft/sec

$v(t_f) = -32t_f = -120$

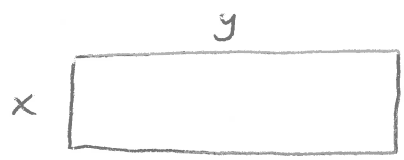
$\Rightarrow t_f = 3.75$ sec

$s(t) = -16t^2 + d$

$0 = -16(3.75)^2 + d$

$\boxed{d = 225 \text{ ft}}$

8. **(10 points)** Find the length and width of a rectangle that has area 75 square feet and a minimum perimeter.



$xy = 75$ (given)

$P = 2x + 2y$ (minimize)

$y = \dfrac{75}{x}$

$P(x) = 2x + \dfrac{150}{x}, \quad x > 0$

$= 2x + 150x^{-1}$

$P'(x) = 2 - \dfrac{150}{x^2} = 0$

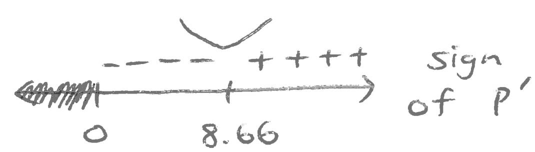
↑

set

$2 = \dfrac{150}{x^2}$

$x^2 = 75 \Rightarrow x = \pm\sqrt{75}$

$x = \sqrt{75} \approx 8.66$



$x = 8.66$ corresponds to a minimum

w/ $x = \sqrt{75}$,

$y = \dfrac{75}{\sqrt{75}} = \sqrt{75} = x$

The rectangle w/ minimum perimeter is a square.