MATH 256
Exam 3
December 14, 2016

Name: Key

DIRECTIONS: Calculators are permitted this exam. However, answers based solely on calculator results are not acceptable (unless it says otherwise). You must show all work to receive full credit. Good luck.

1. **(12 points)** Determine the eigenvalues and associated eigenvectors for $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$.

$A\vec{x} = \lambda\vec{x}$

$(A - \lambda I)\vec{x} = \vec{0}$

set $|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix} = 0$

$-(2-\lambda)(5+\lambda) + 12 = 0$

$-10 + 3\lambda + \lambda^2 + 12 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\lambda = -1, -2$

$\lambda = -1$:

$\left[\begin{array}{cc|c} 3 & -12 & 0 \\ 1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 - 4x_2 = 0$

let $x_2 = t$,

$x_1 = 4t$

$\lambda = -1;\ \vec{x} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\lambda = -2$:

$\left[\begin{array}{cc|c} 4 & -12 & 0 \\ 1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 - 3x_2 = 0$

let $x_2 = t$,

$x_1 = 3t$

$\lambda = -2;\ \vec{x} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

2. **(10 points)** If A has an eigenvalue λ, prove that A^2 has eigenvalue λ^2.

Assume $A\vec{x} = \lambda\vec{x}$

left multiply by A: $A(A\vec{x}) = A(\lambda\vec{x})$

$(AA)\vec{x} = \lambda(A\vec{x})$

$A^2\vec{x} = \lambda(\lambda\vec{x})$

since $A\vec{x} = \lambda\vec{x}$

$A^2\vec{x} = \lambda^2\vec{x}$

so A^2 has eigenvalue λ^2 (w/ same eigenvectors $\vec{x}$). □

3. Given the rule $T(v_1, v_2, v_3) = (v_2 - v_1, v_1 + v_2, 2v_1)$, find

(a) **(5 points)** the image of $\mathbf{v} = (2,3,0)$.

$$T(2,3,0) = (3-2,\ 2+3,\ 2(2))$$
$$= \boxed{(1,5,4)}$$

(b) **(6 points)** the vector whose image is $\mathbf{w} = (-11,-1,10)$.

Set $(v_2 - v_1,\ v_1 + v_2,\ 2v_1) = (-11, -1, 10)$

$$v_2 - v_1 = -11$$
$$v_1 + v_2 = -1$$
$$2v_1 = 10 \rightarrow v_1 = 5$$
$$v_2 = -6$$

v_3 can be any real number

$(5, -6, t)$, where t is real

4. Let T be a linear transformation whose standard matrix is given by $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(a) **(4 points)** Clearly identify m and n for the transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$m = 3$
$n = 2$

(b) **(7 points)** Is T onto $\mathbb{R}^m$? Show detailed reasoning.

onto $\mathbb{R}^3$?

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{array}\right]$$

Will this always have a solution? No, e.g.,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

so not onto $\mathbb{R}^3$.

(c) **(7 points)** Is T a one-to-one mapping? Show detailed reasoning.

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{array}\right]$$

Will this always have at most one solution?

Yes —

Ex: it either has no solution $(c \neq 0)$ or a unique solution (when $c = 0$). Yes, 1-to-1.

5. Prove or disprove that the following transformations T are linear.

(a) **(6 points)** $T: C[0,1] \to \mathbb{R}$ defined by $T(f) = f(0) + 1$

1) Is $T(f+g) = T(f) + T(g)$?

$$\begin{aligned} T(f+g) &= (f+g)(0) + 1 \\ &= f(0) + g(0) + 1 \\ &\neq f(0) + 1 + g(0) + 1 \\ &= T(f) + T(g) \end{aligned}$$

T is <u>not</u> linear

(b) **(6 points)** $T: M_{2\times 2} \to \mathbb{R}$ defined by $T(A) = \det(A)$

1) Is $T(A+B) = T(A) + T(B)$?

$$\begin{aligned} T(A+B) &= |A+B| \\ &\neq |A| + |B| \sim \text{in general} \\ &= T(A) + T(B) \end{aligned}$$

T is <u>not</u> linear

(c) **(6 points)** $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+y, x-y, z)$

1) Is $T(X+Y) = T(X) + T(Y)$?

$$\begin{aligned} T(X+Y) &= T((x_1,y_1,z_1) + (x_2,y_2,z_2)) \\ &= T((x_1+x_2, y_1+y_2, z_1+z_2)) \\ &= ((x_1+x_2)+(y_1+y_2), (x_1+x_2)-(y_1+y_2), z_1+z_2) \\ &= ((x_1+y_1)+(x_2+y_2), (x_1-y_1)+(x_2-y_2), z_1+z_2) = \end{aligned}$$

$$\begin{aligned} &(x_1+y_1, x_1-y_1, z_1) \\ &+ (x_2+y_2, x_2-y_2, z_2) \\ &= T(X) + T(Y) \quad \checkmark \end{aligned}$$

continued

6. **(7 points)** Let $T: V \to W$ be a linear transformation. Prove that if T is one-to-one, then $\ker(T) = \{\mathbf{0}\}$.

Proof: $\ker T = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$

Since T is 1-to-1, $T(\vec{v}) = \vec{0}$ has <u>at most</u> one solution. But $T(\vec{0}) = \vec{0}$ (property of all linear transformations). Thus, $\ker T = \{\vec{0}\}$.

2) Is $T(cX) = cT(X)$?

$$\begin{aligned} T(cX) &= T(cx, cy, cz) \\ &= (cx+cy, cx-cy, cz) \\ &= c(x+y, x-y, z) \\ &= cT(X) \quad \checkmark \end{aligned}$$

so T <u>is</u> linear.

7. Let T be a linear transformation whose standard matrix is given by $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(a) **(7 points)** Find $\ker(T)$ and give its dimension.

$$\left[\begin{array}{cc|c} 5 & -3 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 5 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{x} = \vec{0}$ only

$\ker T = \{\vec{0}\}$

$\dim(\ker T) = 0$

(b) **(7 points)** Find $\operatorname{range}(T)$ and give its dimension.

range T = Col A

By the same sequence seen above, the leading 1s in columns 1 & 2 imply columns 1 & 2 span range T.

range T = span $\left\{ \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} \right\}$

$\dim(\text{range } T) = 2$

(c) **(4 points)** The sum of the two dimensions from parts (a) and (b) total to what number? Why?

$\dim(\ker T) + \dim(\text{range } T) = \dim(\mathbb{R}^2)$

$0 + 2 = 2$ ✓

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8. **(6 points)** Pick <u>one</u> of the statements below and explain why it is true. There is no need for a proof. Instead, give a convincing explanation as to why the statement is valid.

(a) For a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{x}) = A\mathbf{x}$, the $\operatorname{range}(T)$ is equal to the column space of A.

(b) Define $T: M_{nxn} \to M_{nxn}$ by $T(A) = A - A^T$. Then $\ker(T)$ is the set of $n \times n$ symmetric matrices.

ker T is all A where $A - A^T = 0$ (matrix)

or $A = A^T$

If a matrix equals its transpose, we call that matrix symmetric.